Mimar Sinan Fine Arts University * Institute of Science and Technology

A Study on the Chamber Complex

Master's Thesis

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Acknowledgments

Abstract

In this thesis, we define the chamber complex of a given matrix A that is the collection of the chambers we could obtain by moving the halfspaces whose outer normal vectors are given by the rows of A. Moreover, we present an algorithm inspired by the previous works of McMullen, Sturmfels, Henk et. al., Emiris et. al., Beck and others on the chamber of a polytope and the vector partition function. The algorithm works fast for the matrices that have rank two, but it is too slow for most of the matrices of rank three and for matrices of rank more than 3. In order to understand the structure of the chamber complex that can help to improve the algorithm, we consider the chamber complex and its chambers with the toric variety aspect. For a given chamber complex, we examine the toric variety of the normal fan associated to each chamber and also the toric variety of the chamber complex, and present our observations.

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Introduction

In this thesis we define the chamber complex of a given matrix A that is the collection of all chambers we can obtain for A. Moreover, we present an algorithm inspired by the previous works by McMullen, Emiris et. al., Henk et. al., Brion et. al., Sturmfels and others on the chamber of a polytope, and the vector partition function. However, the algorithm we have is too slow for most of the cases in dimension 3 and for dimension more than 3. In order to understand the structure of the chamber complex better, we consider the toric variety associated to the chambers of a given chamber complex, and the toric variety of a given chamber complex. In this thesis we present some examples and observations that can be helpful for the future works on improving the algorithm we have.

A polytope $P \in \mathbb{R}^d$ is the bounded intersection of finitely many closed halfspaces, i.e., the solution set of the system $Ax \leq b$ of inequalities for some matrix $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$. The Minkowski sum of two polytopes P and Q is defined as

$$P + Q = \{x + y : x \in P, y \in Q\}.$$

The chamber (or type cone, or Minkowski cone) C_P of the polytope P was first defined by McMullen in 1973 in [10] which presents a way to answer the question asking, for a given polytope $P \subset \mathbb{R}^d$, whether there exist two polytopes $R, S \subset \mathbb{R}^d$ (called summands), that are not obtained by dilating or shrinking or translating P, and whose Minkowski sum $R + S$ gives P. If such polytopes exist, then P is called decomposable, if not, P is called indecomposable. The chamber C_P is the cone of all vectors $c \in \mathbb{R}^m$ for which the solution set P_c of $Ax \leq c$ gives a polytope that is strongly combinatorially equivalent to P, i.e., they have the same normal fan (collection of the cones generated by the normal vectors of the faces). McMullen stated that a polytope P is indecomposable if and only if the chamber C_P of P is one dimensional. Moreover, the generating rays of the chamber give information about the indecomposable summands of P, i.e., for a point $d \in \mathbb{R}^m$, taken from one of the generating rays of C_P, the polytope given by the solution set of $Ax \leq d$ is an indecomposable summand of P. Since then, Emiris et. al.[6], Henk et. al. [8], Brion et. al.[3], Beck [1] and others worked on the chamber and its applications to integer programming and combinatorics. Sturmfels in 1994 [11]proved that for each chamber C_P, there exists a vector partition function that gives the number of lattice points of the polytope P_c for any point c taken from C_P. The vector partition function became a very useful tool in Ehrhart theory as the vector partition function of a chamber C_P is indeed an Ehrhart polynomial.

A given matrix $A \in \mathbb{R}^{m \times d}$ of rank d corresponds to the set of outer normal vectors of the defining half spaces. For any vector $b \in \mathbb{R}^m$, the solution set $P_b = \{x \in \mathbb{R}^d : Ax \leq b\}$ is a polytope. Replacing the right hand side vector b with another vector $c \in \mathbb{R}^m$ corresponds

to moving the halfspaces along the directions given by A. Each such replacement gives another polytope that may have different combinatorial type, i.e., they have different relations between their faces. Being given a polytope indeed corresponds to having a fixed matrix A and a right hand side vector b that fixes the polytope P_b, i.e., the combinatorial type. The chamber C_P is the cone of those vectors b, such that the movement of hyperplanes preserves the combinatorial type.

In this thesis, we present the chamber complex defined as, for a given matrix $A \in \mathbb{R}^{m \times d}$, the collection of all the chambers one can obtain by changing the right hand side vector in the system of inequalities $Ax \leq b$. In particular, there is one chamber for each combinatorial type one can obtain by moving the halfspaces whose directions are given by A. When the chamber complex of a given matrix A has only one top-dimensional chamber, the chamber complex of A coincides with the closure of the chamber of the polytope P whose chamber is the top-dimensional cone in the chamber complex. However, we observed that for some matrices A, there can be more than one top-dimensional chambers in the chamber complex of A (See the Example 2.2.1.2). We describe an algorithm that computes the chamber complex of a given matrix A (See Algorithm 2.1). The algorithm we describe is too slow for computing the chamber complex of most of the matrices of rank 3 and for matrices of rank bigger than 3, although the algorithm from previous works would be fast. We also know that the algorithm from previous works is not enough to get the chamber complex when it has more than one top dimensional chambers as we cannot realize the chamber complex as the closure of the chamber of a polytope.

We have the following approach in order to understand the structure of the chamber complex better and have some observations that can help to improve the algorithm we have with a future work.

One of the classical objects of algebraic geometry is toric varieties. We mentioned above that for each chamber in the chamber complex, there is one associated combinatorial type. Moreover, we have that for each chamber C, there is one associated normal fan such that all polytopes P_b for b in C have the same normal fan. Let us consider the toric varieties of the normal fans associated to the chambers. When we move in the chamber complex from one chamber to another, the toric varieties of the associated normal fans change and when two chambers have a common face, the toric varieties have a common sub-variety. In Chapter 3, we focus on the change on of the toric varieties associated to the chambers of a given chamber complex and provide theorems on the change, when the chambers are neighbor in specific conditions (See Theorem 3.0.0.2 and Theorem 3.0.0.3).

As each chamber itself is a strictly convex rational polyhedral cone, we can also consider the toric variety of the chamber complex. Moreover, every toric variety X_Σ has a natural torus action on it, and the number of fixed points is the number of top dimensional cones of the fan Σ. We have the observation that the number of top dimensional chambers of the chamber complex C_A of a given matrix A is the number of the fixed points of the torus action $\gamma : T \times X_{C_A} \to X_{C_A}$ where X_{C_A} is the toric variety of the chamber complex C_A (See Theorem 3.0.1.2).

Chapter 1

Preliminaries

1.1 Polyhedral Geometry

In this section, we give the necessary definitions to understand the theory of the chamber and the chamber complex.

Let $A = \{a_1, \cdots, a_m\} \in \mathbb{R}^{m \times d}$ be an arbitrary but fixed matrix such that $rank(A) = d$. Let H_i be a closed half-space for $i \in \{1, \cdots, m\}$ which can be written as

$$H_i = \left\{ x \in \mathbb{R}^d : \langle a_i, x \rangle \leq b_i \right\}$$

for some $b_i \in \mathbb{R}$ such that a_i is the outer normal vector of H_i. A polytope P_b is the bounded intersection of finitely many (in this setting "m") closed half spaces, i.e.,

$$P_b = \bigcap_{i=1}^m H_i = \bigcap_{i=1}^m \left\{ x \in \mathbb{R}^d : \langle a_i, x \rangle \leq b_i \right\}.$$

Thus, P_b is the solution set of the system of inequalities and can be expressed as

$$P_b = \left\{ x \in \mathbb{R}^d : Ax \leq b \right\}$$

where $b = (b_1, \cdots, b_m)$. We can add slack variables and turn the system of inequalities into a system of equalities and express P_b as

$$P_b = \left\{ x \in \mathbb{R}^d : [A, I] \begin{bmatrix} x \\ y \end{bmatrix} = b, \ y \geq 0 \right\}.$$

Being given the matrix $A \in \mathbb{R}^{m \times d}$ of rank d corresponds to having a fixed set of outer normal vectors of half spaces $H_i = \left\{ x \in \mathbb{R}^d : \langle a_i, x \rangle \leq b_i \right\}$ for some $b_i \in \mathbb{R}$ for $i \in [m]$. Furthermore, changing the components b_i of the right hand side vector b corresponds to moving the half spaces H_i around. When we move the half spaces by inserting a different vector b', the intersection $\bigcap_{i=1}^m H_i$ changes and some half spaces may become redundant for the polytope $P_{b'} = \bigcap_{i=1}^m H_i$.

Example 1.1.0.1. *Consider $A = \{a_1 = (1, 0), a_2 = (0, 1), a_3 = (-1, 0), a_4 = (0, -1), a_5 = (1, 1)\}$ that is the set of normal vectors of the closed-halfspaces that are illustrated by Figure 1.1a,*
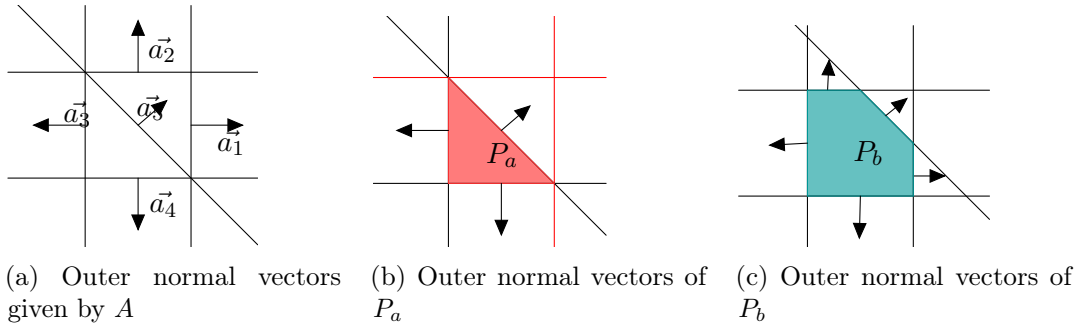
(a) Outer normal vectors given by A

(b) Outer normal vectors of P_a

(c) Outer normal vectors of P_b

Figure 1.1: Outer normal vectors of halfspaces and their appearance on polytopes

- *For $a = (1, 1, 0, 0, 1)$, the halfspaces $H_1 = \{x \in \mathbb{R}^d : \langle a_1, x \rangle \leq b_1\}$ and $H_2 = \{x \in \mathbb{R}^d : \langle a_2, x \rangle \leq b_2\}$ become redundant for $P_a = \bigcap_{i=1}^{5} H_i = \bigcap_{i=3}^{5} H_i$ which is demonstrated in Figure 1.1b.*

- *When we consider Figure 1.1c , we see that there is no redundant halfspace for*

$$P_b = \bigcap_{i=1}^{5} H_i = \left\{ x \in \mathbb{R}^d : Ax \leq b, \right\} \text{ where } b = (2, 2, 0, 0, 3).$$

Figure 1.2 illustrates some polytopes that can be obtained by moving H_i's. We can get a square or triangle or the other polytope that appears in the figure.
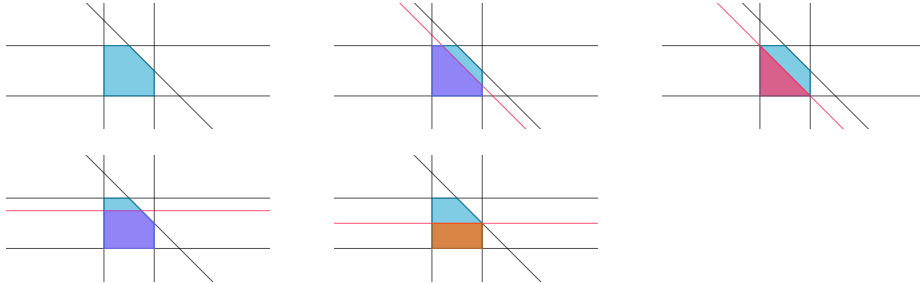
Figure 1.2: Different polytopes that can be obtained by moving halfspaces

Note on the notation: Note that any d-dimensional polytope P can be expressed as

$$P = \left\{ x \in \mathbb{R}^d : Ax \leq b \right\}$$

for some matrix $A \in \mathbb{R}^{m \times d}$ and some vector $b \in \mathbb{R}^m$.

For polytopes, we use the notation with different letters (P, Q etc.) when the polytopes do not need to be defined by the same matrix. We use the notation with sub indexes

$(P_b, P_c$ etc.) when it is important for us that the polytopes are defined by the same matrix A and different right hand side vectors that we put as index.

Definition 1.1.0.2. *Two polytopes P and Q are called combinatorially equivalent if they have the same face lattices.*

Example 1.1.0.3. *The polytopes below have the following face lattices.*
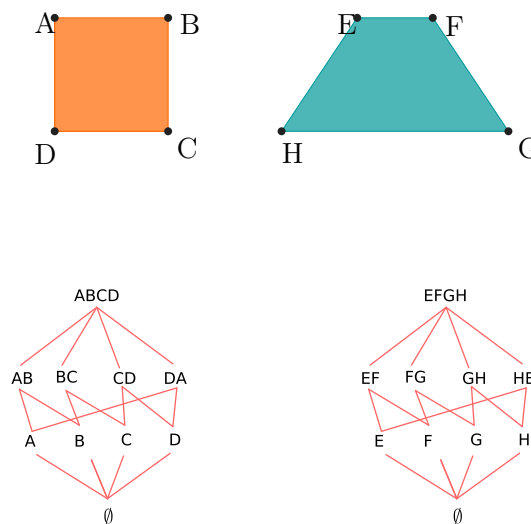
Figure 1.3: Combinatorially equivalent polytopes

Since they have the same face lattice, i.e., they have the same relations between their faces, they are combinatorially equivalent.

Definition 1.1.0.4. *Given a $d-$dimensional polytope P, the support function for P is defined as*

$$h(P, \cdot) : \mathbb{R}^d \to \mathbb{R}, \quad h(P, v) = sup\left\{\langle v, x\rangle : x \in P\right\}.$$

A hyperplane

$$H(P, v) = \left\{x \in \mathbb{R}^d : \langle x, v\rangle = h(P, v)\right\}$$

for $v \in \mathbb{R}^d \setminus \{0\}$ is called a supporting hyperplane of P.

Example 1.1.0.5. *$H_1, \ldots, H_5$ are supporting hyperplanes of P illustrated in Figure 1.4.*

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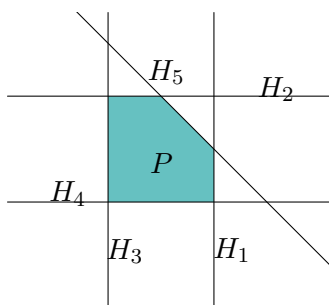
Figure 1.4: Supporting hyperplanes of the polytope P

Definition 1.1.0.6. *The intersection of a polytope $P \in \mathbb{R}^d$ with a supporting hyperplane $H(P, v)$ is called a face of P. A $(d-1)$-dimensional face of P is called a facet.*

Example 1.1.0.7. *In Figure 1.5, $F_1, \ldots, F_5$ are facets of P as each F_i for $i \in [5]$ is intersection of H_i with P. Moreover, $v_{34}, v_{23}, v_{25}, v_{15}, v_{14}$, which are intersections of two hyperplanes with the polytope, are one dimensional faces of P.*
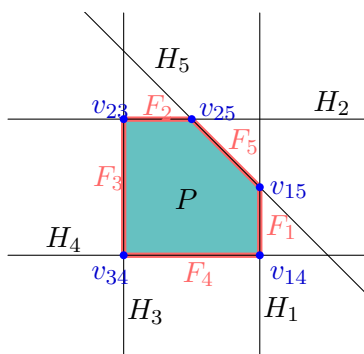
Figure 1.5: Faces of the polytope P

Definition 1.1.0.8. *The normal cone $N(F, P)$ of a face F of a polytope P is the set of all vectors $v \in \mathbb{R}^d$ for which the supporting hyperplane $H(P, v)$ contains F, that is*

$$N(F, P) = \{v \in \mathbb{R} \dot{.} F \subseteq H(P, v) \cap P\}.$$

The collection of all normal cones of all faces of P is called the normal fan $N(P)$ of P.

$$N(P) = \{N(F, P) : F \text{ is a face of } P\}.$$

In other words, for a given polytope P, the face cone $N(F, P)$ of a facet F of P is the cone generated by the normal vector of F. Since we can express the lower dimensional faces of P by the intersection of some facets, the normal cone of a lower dimensional face

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$V = F_1 \cap \ldots \cap F_n$ of P, where F_i's are some facets of P, is the cone generated by the normal vectors of $F_1, \ldots, F_n$.

Example 1.1.0.9. *In this example, we illustrate the normal cones of faces and normal fans of the polytopes P and Q that are given in Figure 1.6.*
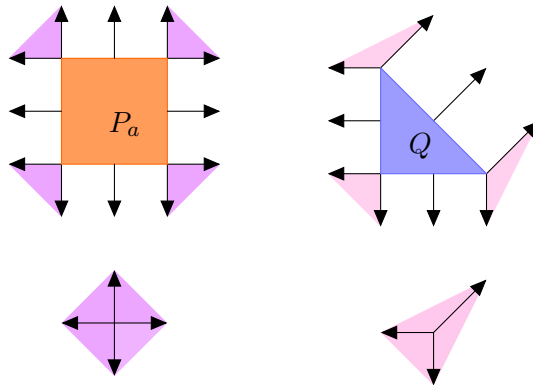
Figure 1.6: Normal fans of the polytopes P and Q

The cones $N(P)$ contains			*The cones $N(Q)$ contains*	
Cones	*Generating rays*			
$N(F_1, P)$	$(0,1)$			
$N(F_2, P)$	$(1,0)$		*Cones*	*Generating rays*
$N(F_3, P)$	$(-1,0)$		$N(F_1, Q)$	$(-1,0)$
$N(F_4, P)$	$(0,-1)$		$N(F_2, Q)$	$(0,-1)$
$N(V_{12}, P)$	$(0,1),(1,0)$		$N(F_3, Q)$	$(1,1)$
$N(V_{13}, P)$	$(0,1),(-1,0)$		$N(V_{12}, Q)$	$(-1,0),(0,-1)$
$N(V_{24}, P)$	$(1,0),(0,-1)$		$N(V_{13}, Q)$	$(-1,0),(1,1)$
$N(V_{43}, P)$	$(0,-1),(-1,0)$		$N(V_{23}, Q)$	$(0,-1),(1,1)$

For a given matrix $A = \{a_1, \cdots, a_m\} \in \mathbb{R}^{m \times d}$, i.e., set of outer normal vectors, and for any non-empty polytope $P_b = \{x \in \mathbb{R}^d : Ax \leq b\}$, it is clear from the definition of the normal fan of a polytope, that the set of generating rays of the normal fan $N(P_b)$ is a subset of A, and it is exactly A, when there is no redundant halfspace for P_b.

Definition 1.1.0.10. *Two polytopes P, Q are called strongly combinatorially equivalent if and only if they have the same normal fan.*

Example 1.1.0.11. *Consider Figure 1.7. One can observe that P and Q are neither combinatorially equivalent, nor strongly combinatorially equivalent while P and R are combinatorially equivalent, but not strongly combinatorially equivalent.*

Figure 1.7: Polytopes P, Q and R, and their normal fans

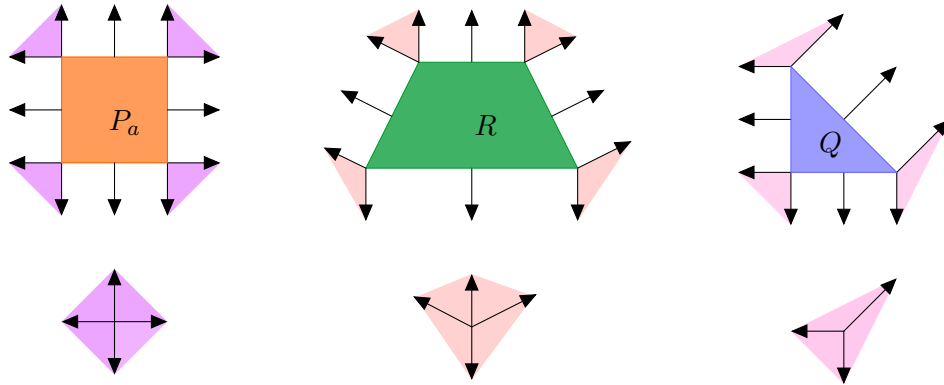| *The cones $N(P)$ contains*		*The cones $N(R)$ contains*		*The cones $N(Q)$ contains*	
Cones	*Generating rays*	*Cones*	*Generating rays*		
$N(F_1, P)$	$(0,1)$	$N(F_1, R)$	$(0,1)$	*Cones*	*Generating rays*
$N(F_2, P)$	$(1,0)$	$N(F_2, R)$	$(2,1)$	$N(F_1, Q)$	$(-1,0)$
$N(F_3, P)$	$(-1,0)$	$N(F_3, R)$	$(0,-1)$	$N(F_2, Q)$	$(0,-1)$
$N(F_4, P)$	$(0,-1)$	$N(F_4), R$	$(-2,1)$	$N(F_3, Q)$	$(1,1)$
$N(V_{12}, P)$	$(0,1),(1,0)$	$N(V_{12}, R)$	$(0,1),(2,1)$	$N(V_{12}, Q)$	$(-1,0),(0,-1)$
$N(V_{13}, P)$	$(0,1),(-1,0)$	$N(V_{23}, R)$	$(2,1),(0,-1)$	$N(V_{13}, Q)$	$(-1,0),(1,1)$
$N(V_{24}, P)$	$(1,0),(0,-1)$	$N(V_{34}, R)$	$(0,-1),(-2,1)$	$N(V_{23}, Q)$	$(0,-1),(1,1)$
$N(V_{43}, P)$	$(0,-1),(-1,0)$	$N(V_{14}, R)$	$(0,1),(-2,1)$		

When we consider the normal fans of the polytopes K and L given by Figure 1.8, we see that their normal fans are the same, such that K and L are strongly combinatorially equivalent.

The cones $N(K)$ contains		*The cones $N(L)$ contains*	
Cones	*Generating rays*	*Cones*	*Generating rays*
$N(F_1, K)$	$(0,1)$	$N(F_1, L)$	$(0,1)$
$N(F_2, K)$	$(1,0)$	$N(F_2, L)$	$(1,0)$
$N(F_3, K)$	$(-1,0)$	$N(F_3, L)$	$(-1,0)$
$N(F_4, K)$	$(0,-1)$	$N(F_4, L)$	$(0,-1)$
$N(F_5, K)$	$(1,1)$	$N(F_5, L)$	$(1,1)$
$N(V_{13}, K)$	$(0,1),(-1,0)$	$N(V_{13}, L)$	$(0,1),(-1,0)$
$N(V_{15}, K)$	$(0,1),(1,1)$	$N(V_{15}, L)$	$(0,1),(1,1)$
$N(V_{52}, K)$	$(1,1),(1,0)$	$N(V_{52}, L)$	$(1,1),(1,0)$
$N(V_{43}, K)$	$(0,-1),(-1,0)$	$N(V_{43}, L)$	$(0,-1),(-1,0)$
$N(V_{24}, K)$	$(1,0),(0,-1)$	$N(V_{24}, L)$	$(1,0),(0,-1)$
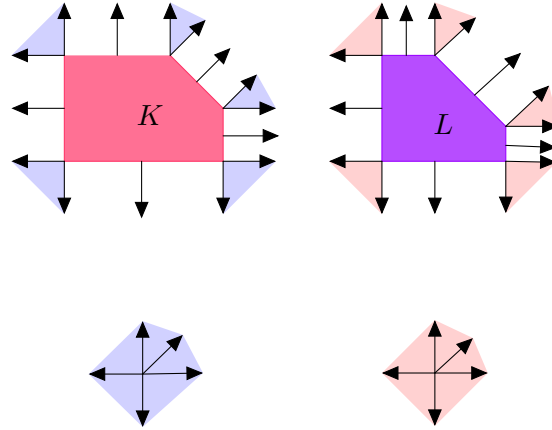
Figure 1.8: Normal fans of the strongly combinatorially equivalent polytopes K and L

Definition 1.1.0.12. *The Minkowski sum of two polytopes $P, Q \subset \mathbb{R}^d$ is defined as follows;*

$$P + Q = \{x + y : x \in P, y \in Q\}.$$

Example 1.1.0.13. *The Minkowski sum of given polytopes*

$$P = \left\{ x \in \mathbb{R}^2 : A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$and\ Q = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} x \leq c = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$is\ P + Q = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 3 \end{pmatrix} \right\}\ and\ illustrated\ in\ Figure\ 1.9.$$

Figure 1.9: Minkowski sum of two polytopes P and Q

We can also add redundant halfspaces and define P and Q with the same matrix such that

$$P = \left\{ x \in \mathbb{R}^2 : A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} x \leq b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$$\text{and } Q = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} x \leq c = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$\text{Thus, } P + Q = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 3 \end{pmatrix} \right\}.$$
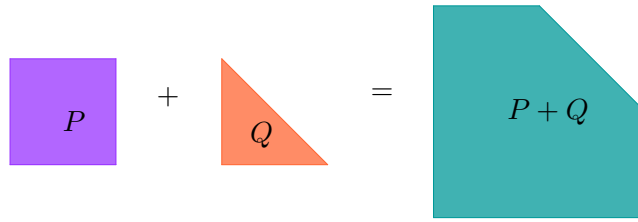
The Minkowski sum with redundant hyperplanes is demonstrated in Figure 1.10. After taking the sum, the redundant hyperplanes become non-redundant and support the polytope $P + Q$.
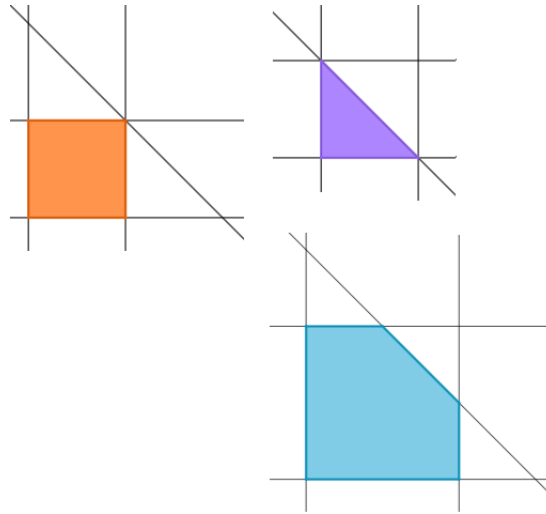
Figure 1.10: Illustration of the Minkowski sum with the same hyperplanes

Definition 1.1.0.14. *Two polytopes $P, Q \subset \mathbb{R}^d$ are called homothetic if and only if $P = \lambda Q + t$ for some $\lambda \in \mathbb{R}_{\geq 0}$ and $t \in \mathbb{R}^d$.*

In other words, when we translate or dilate a polytope P, the combinatorial type of the P does not change and we obtain the polytope which is homothetic to P.

Example 1.1.0.15. *Consider the polytope* $P = \left\{ x \in \mathbb{R}^2 : A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$

P, $P + (0,1)$ and $\frac{3}{2}P$ have the same combinatorial type and they are homothetic as illustrated in the figure 1.11.
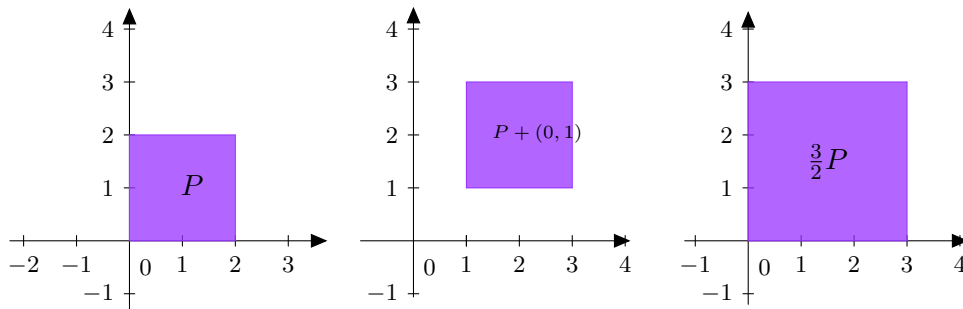
Figure 1.11: Homothetic Polytopes

Definition 1.1.0.16. *A polytope $P \subset \mathbb{R}^d$ is called decomposable if there are polytopes Q, R, that are not homothetic to P, such that*

$$P = Q + R.$$

If there are no such polytopes Q, R, then P is called indecomposable. Q and R are called summands of P.

Example 1.1.0.17. *The polytope P in Figure 1.12 is decomposable as there are non-homothetic polytopes Q, R whose Minkowski sum is P, where*

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 3 \end{pmatrix} \right\},$$

$$Q = \left\{ x \in \mathbb{R}^2 : A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} x \leq b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$$\text{and } R = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} x \leq c = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$
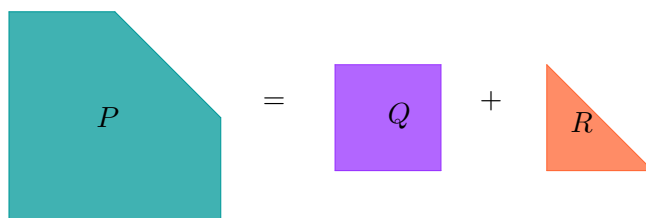
Figure 1.12: Minkowski decomposition of the polytope P

Consider a decomposition of the polytope K given in Figure 1.13. The polytope $\frac{1}{2}K$ that appears as a summand is homothetic to K, so this is not a non-homothetic decomposition. In fact, K is indecomposable.
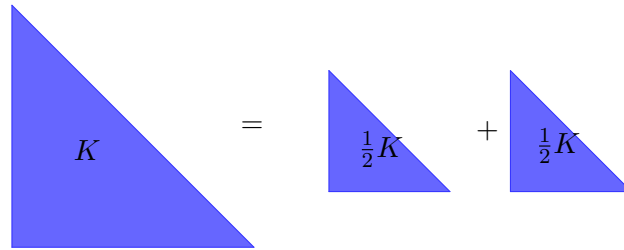
Figure 1.13: Homothetic decomposition of the polytope K

Remark 1.1.0.18. *Given a polytope $P \subset \mathbb{R}^d$, for any polytope Q that is homothetic to P, $N(Q) = N(P)$, i.e., they are strongly combinatorially equivalent. However, a polytope R, that is strongly combinatorially equivalent to P, does not need to be homothetic to P.*

Example 1.1.0.19. *Consider polytopes* $P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}$,

$$Q = 2P \ \text{and} \ R = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \right\}$$

illustrated in figure 1.14.
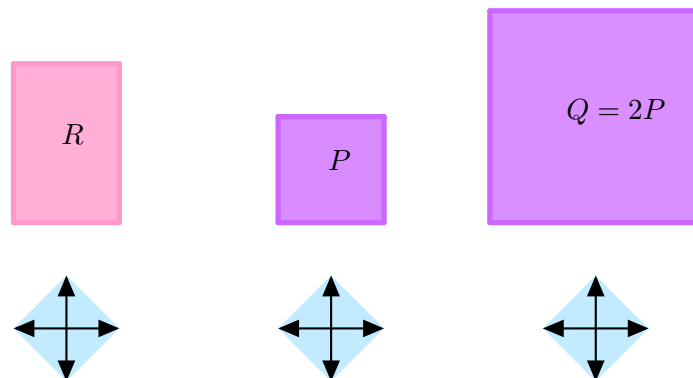
Figure 1.14: Relation between being homothetic and strongly combinatorially equivalent

Polytopes P, Q and R are strongly combinatorial equivalent since they have the same normal fan. However, the polytope R is not homothetic to P while the polytope Q is.

Definition 1.1.0.20. *A polyhedral complex is a collection Δ of polyhedral cones satisfying the following.*

- *If $K \in \Delta$ and F is a face of K, then $F \in \Delta$,*

- *If $K, L \in \Delta$, $K \cap L$ is either empty or a face of K and a face of L.*

1.2 Toric Varieties

In this section, we give the definitions that are needed to understand the relation between the toric variety of a fan and a chamber of a chamber complex.

Definition 1.2.0.1. *Let $R \subset \mathbb{C}[x_1, \ldots, x_m]$ be a polynomial ring. For a given ideal $I \subset R$, the affine variety associated to I is*

$$V(I) = \{p \in \mathbb{C}^m : f(p) = 0 \text{ for all } f \in I\}.$$

Moreover, for a given variety $V \subset \mathbb{C}^m$, the ideal associated to V is

$$I(V) = \{f \in R : f(p) = 0 \text{ for all } p \in V\}.$$

The coordinate ring associated to V is $\mathbb{C}[V] = R/I(V)$.

Theorem 1.2.0.2 (Hilbert's Nullstellensatz)**.** *With the notation used above,*

$$I(V(I)) = \sqrt{I} = \left\{f \in R : f^l \in I \text{ for some } l \geq 1\right\}.$$

Let us recall some facts on the coordinate rings. $\mathbb{C}[V]$ is an integral domain if and only if $I(V)$ is a prime ideal if and only if V is irreducible, i.e., if $V = A \cup B$ where A, B are affine varieties, $V = A$ or $V = B$.

Definition 1.2.0.3. *A polyhedral cone $\sigma = \{x \in \mathbb{R}^m : x = \sum_{i=1}^r v_i \lambda_i, \ \lambda_i \in \mathbb{R}_{\geq 0}\} = Pos(v_1, \ldots, v_m)$ with generators $\{v_1, \ldots, v_m\}$ is called a rational cone (lattice cone) if the generators of σ belong to a lattice $N \cong \mathbb{Z}^m \subset \mathbb{R}^m$.*
A cone σ is called strictly convex if it contains no line going through the origin, i.e., $\sigma \cap (-\sigma) = \{0\}$.

Definition 1.2.0.4. *A strictly convex polyhedral cone $\sigma \subset \mathbb{R}^m$ is called smooth or regular if the minimal generators of σ spans $\mathbb{Z}^m$.*

Definition 1.2.0.5. *The dual cone σ^* of a cone σ is defined as follows:*

$$\sigma^* = \{y \in (\mathbb{R}^m)^* : \langle y, x \rangle \geq 0, \forall x \in \sigma\}.$$

Example 1.2.0.6. *Consider the cone $\sigma = \{x \in \mathbb{R}^2 : x = (2,2)\lambda_1 + (-4,2)\lambda_2, \ \lambda_i \in \mathbb{R}_{\geq 0}\}$ is illustrated in Figure 1.15.*
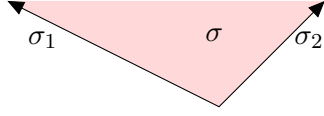
Figure 1.15: The cone σ

In order to find the dual cone σ^* of σ, we first find the dual cones of σ_1, σ_2 as demonstrated in Figure 1.16.
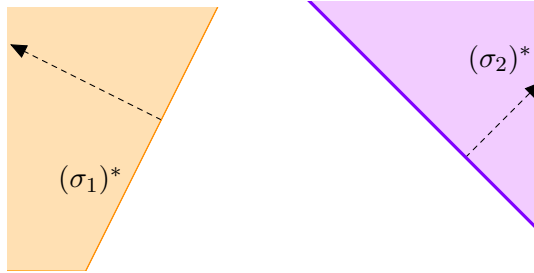
Figure 1.16: Dual cones of σ_1 and σ_2

The dual cone σ^* is the intersection of the duals of the generating rays. Namely,

$$\sigma^* = (\sigma_1)^* \cap (\sigma_2)^* = \langle e_1^* + 2e_2^*, -e_1^* + e_2^* \rangle$$

illustrated in Figure 1.17.
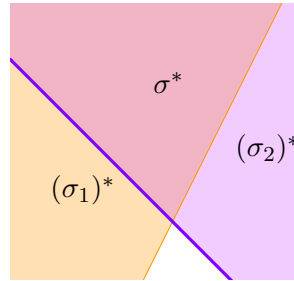
Figure 1.17: The cone σ^*

Definition 1.2.0.7. *If σ is a lattice cone (rational), i.e., its generators are contained in a lattice $N \cong \mathbb{Z}^m \subset \mathbb{R}^m$, then σ^* is also a rational cone such that its generators belong to the lattice $M = Hom_{\mathbb{Z}}(N, \mathbb{Z}) \cong \mathbb{Z}^m \subset (\mathbb{R}^m)^*$ (dual lattice of N).*

Definition 1.2.0.8. *Let σ be a cone and let $u \in \sigma^* \cap M$ where M is a lattice in $(\mathbb{R}^m)^*$. A face τ of σ is defined as*

$$\tau = \sigma \cap u^\perp = \{x \in \sigma : \langle u, x \rangle = 0\}.$$

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Example 1.2.0.9. *Consider the cone σ generated by $2e_1 + 2e_2$ and $-4e_1 + 2e_2$ and consider the cone σ_1 generated by $-4e_1 + 2e_2$ as a face of σ.*
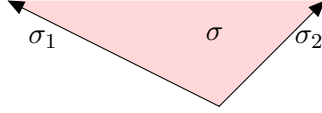
The vector $u = 2e_1 + 4e_2 \in \sigma^*$ satisfies that $\sigma_1 = \sigma \cap u^\perp$.
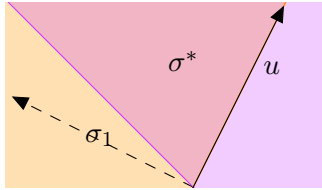
Lemma 1.2.0.10. *Gordan's Lemma*

If σ is a polyhedral lattice (rational) cone, then $\sigma \cap N$ is a finitely generated monoid satisfying the simplification law ($s + t = s' + t \Rightarrow s = s'$ for s, s' and t are in $S = \sigma \cap N$).

When we consider the dual cone σ^* of σ and apply the lemma, we have that $\sigma^* \cap M$ denoted by S_σ is a finitely generated monoid.

Example 1.2.0.11. *Consider the cone σ_1 generated by $-2e_1 + e_2$.*
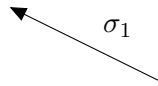
The intersection $S_{\sigma_1} = (\sigma_1)^* \cap M \cong \mathbb{Z}^2 \in \mathbb{R}^2$ is the monoid generated by

$$e_1^* + 2e_2^*, \ -e_1^*, \ e_2^*, \ -e_1^* - e_2^*, \ and \ -e_1^* - 2e_2^*.$$
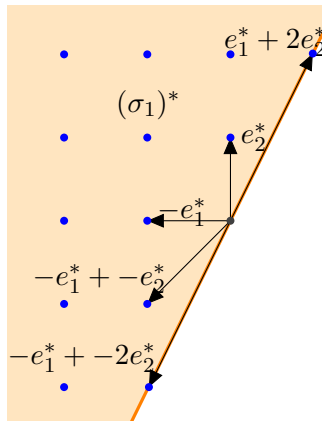
Now, let us consider the ring $\mathbb{C}[z_1, \ldots, z_m, z_1^{-1}, \ldots, z_m^{-1}] = \mathbb{C}[z, z^{-1}]$ which is called the ring of Laurent polynomials. A Laurent monomial is of the form $\lambda z^a = \lambda z_1^{a_1} \cdots z_m^{a_m}$ where $\lambda \in \mathbb{C}^*$ and $a = (a_1, \ldots, a_m)$ is in $\mathbb{Z}^m$.

Remark 1.2.0.12. *The mapping*

$$\phi : \mathbb{Z}^m \to \mathbb{C}[z, z^{-1}]$$

$$a = (a_1, \ldots, a_m) \to z_1^{a_1} \cdots z_m^{a_m}$$

is an isomorphism between the group $(\mathbb{Z}^m, +)$ *and the group* $(LM, \cdot)$ *where* LM *is the set of monic Laurent monomials (coefficients are 1).*

Definition 1.2.0.13. *The support of a polynomial* $f = \sum_{finite} \lambda_a z^a$ *is defined as*

$$supp(f) = \{a \in \mathbb{Z}^m : \lambda_a \neq 0\}$$

Proposition 1.2.0.14. *For a rational polyhedral cone* σ, *the ring*

$$R_\sigma = \left\{ f \in \mathbb{C}[z, z^{-1}] : supp(f) \subset \sigma^* \cap M = S_\sigma \right\}$$

is a finitely generated monomial algebra which is generated by the monomials $\phi(s_{\sigma i})$, *where* $s_{\sigma i}$ *are generators of* S_σ *and*

$$\phi : \mathbb{Z}^m \to \mathbb{C}[z, z^{-1}], \ a = (a_1, \ldots, a_m) \to z_1^{a_1} \cdots z_m^{a_m}.$$

In order to associate the ring R_σ to a coordinate ring, let us consider the ring $\mathbb{C}[x_1, \cdots, x_k]$ of polynomials in k variables over $\mathbb{C}$. A finitely generated monomial algebra R_σ can be written as $\mathbb{C}[x_1 \cdots, x_k]/I$ for some k and the ideal I where the ideal is generated by the relations between the monomials in R_σ.

Example 1.2.0.15. *From the previous example, we know that the monoid* S_{σ_1} *for the cone* $\sigma_1 = Pos(-2e_1 + e_2)$ *is generated by*

$$e_1^* + 2e_2^*, \ -e_1^*, \ e_2^*, \ -e_1^* - e_2^*, \ and \ -e_1^* - 2e_2^*.$$

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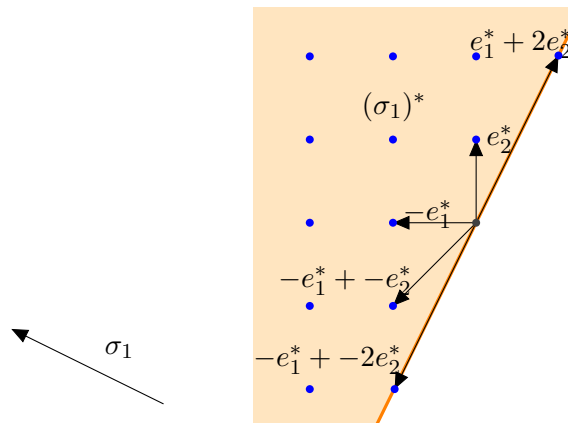
By the mapping $\phi : \mathbb{Z}^2 \to \mathbb{C}[z, z^{-1}]$,

$$\phi((1,2)) = z_1 z_2^2, \; \phi((0,1)) = z_2, \; \phi((-1,0)) = z_1^{-1},$$
$$\phi((-1,-1)) = z_1^{-1} z_2^{-1} \; and \; \phi((-1,-2)) = z_1^{-1} z_2^{-2}.$$

Thus, R_{σ_1} is generated by the monomials

$$m_1 = z_1 z_2^2, \; m_2 = z_2, \; m_3 = z_1^{-1}, \; m_4 = z_1^{-1} z_2^{-1} \; and \; m_5 = z_1^{-1} z_2^{-2}.$$

We can express R_{σ_1} as

$$R_{\sigma_1} = \mathbb{C}[m_1, \ldots, m_5] = \mathbb{C}[x_1, \ldots x_5]/I_{\sigma_1}.$$

Moreover, the generating monomials have the following relations:

$$m_4 m_3 = m_5, \; m_2 m_4 = m_3, \; m_1 m_5 = 1.$$

Thus, the ideal I_{σ_1} is generated by the relations $x_4 x_1 = x_2$, $x_1 x_4^2 = x_3$, $x_1 x_5 = 1$.

Definition 1.2.0.16. *The toric variety X_σ of a strictly convex, rational polyhedral cone σ is defined as*

$$X_\sigma = \left\{ x \in \mathbb{C}^k : f(x) = 0, \; f \in I_\sigma \right\} = Specmax(R_\sigma)$$

where $Specmax(R_\sigma)$ is the set of all maximal ideals of R_σ and $R_\sigma = \mathbb{C}[X_\sigma]$.

Example 1.2.0.17. *In the Example 1.2.0.15, I_{σ_1} was the ideal generated by the relations $x_4 x_1 = x_2$, $x_1 x_4^2 = x_3$, $x_1 x_5 = 1$. The toric variety X_{σ_1} for the cone $\sigma_1 = Pos(-2e_1 + e_2)$ is the following set:*
$$X_{\sigma_1} = \left\{ x \in \mathbb{C}^5 : f(x) = 0, \; f \in I_{\sigma_1} \right\}$$

Since we mention about the fixed points of the torus action on variety in the chapter 3, let us give another definition of a toric variety.

Definition 1.2.0.18. *A toric variety X is an irreducible affine variety containing a torus $T \cong (\mathbb{C}^*)^m$ as a Zariski open subset such that the action of T on itself extends to an action $\gamma : T \times X \to X$ of T on X.*

Please note that a torus T is an affine toric variety that is isomorphic to $(\mathbb{C}^*)^m$ which is a group under component-wise multiplication.

Theorem 1.2.0.19 (Cox, Little, Schenk [4])**.** *Definition 1.2.0.16 with the given construction and Definition 1.2.0.18 are equivalent.*

For more information on the torus action and toric varieties, you may check [4], [9], [7].

1.2.1 Toric Varieties and Polytopes

In the previous section, we defined the toric variety of a cone. In this section we consider the toric variety of a fan, that is a collection of polyhedral cones satisfying some conditions. Then we observe the toric variety of the normal fan of a polytope, which helps us to illustrate the connection of the chamber complex with toric varieties.

For a given cone σ, when we consider a face $\sigma_1 = \sigma \cap u^\perp$ of σ with $u \in S_\sigma$, we see that the monoid

$$S_{\sigma_1} = \{S_\sigma + t.(-u) : t \in \mathbb{Z}_{\geq 0}\}$$

is obtained by adding one more generator. Namely, for S_σ and R_σ with r generators, S_{σ_1} and R_{σ_1} have $r+1$ generators. Thus, we can obtain $R_\sigma = \mathbb{C}[x_1, \ldots, x_r]/I_\sigma$ from $R_{\sigma_1} = \mathbb{C}[x_1, \ldots, x_{r+1}]/I_{\sigma_1}$ by adding an additional relation.

Remark 1.2.1.1. *For a rational polyhedral cone σ and a face σ_1 of σ with $r, r+1$ generators respectively, there is an identification $X_{\sigma_1} \cong X_\sigma \backslash (x_{r+1} = 0)$*

For cones σ and α having a common face σ_1, we can glue σ and α on their common part via the map,

$$\psi_\sigma^\alpha : X_\sigma \setminus (x_{r+1} = 0) \cong X_{\sigma_1} \cong X_\alpha \setminus (y_{k+1} = 0).$$

Definition 1.2.1.2. *A fan Σ in $\mathbb{R}^d$ is a finite union of cones such that:*

- *Every cone in Σ is strictly convex, polyhedral, lattice cone,*

- *Every face of a cone of Σ is a cone of Σ,*

- *If σ_1, σ_2 are cones in Σ, then $\sigma_1 \cap \sigma_2$ is a common face of σ_1 and σ_2.*

  From the last two conditions, one may deduce that a fan Σ is a polyhedral complex. Furthermore, the support of a fan Σ is $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$.

  A fan $\Sigma \subset \mathbb{R}^d$ is called complete if it spans $\mathbb{R}^d$. For example, for a polytope P, the normal fan $N(P)$ is complete.

In order to compute the toric variety of a fan Σ, we first find the toric variety of the maximal cones of Σ and then glue them along the common faces of the cones with a gluing map ψ.

Example 1.2.1.3. *Consider the fan Σ containing the following cones as illustrated in Figure 1.18.*

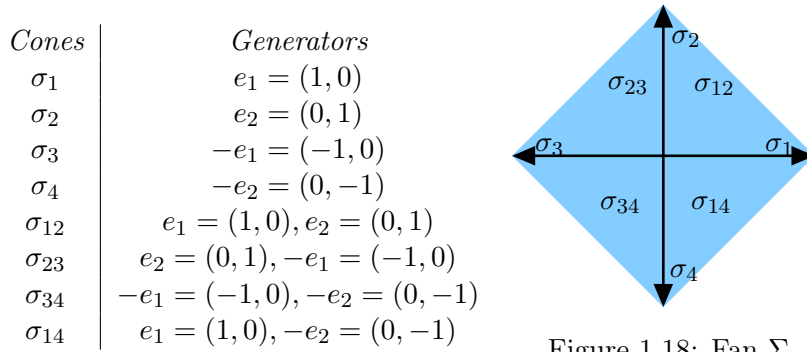| Cones	Generators
σ_1	$e_1 = (1, 0)$
σ_2	$e_2 = (0, 1)$
σ_3	$-e_1 = (-1, 0)$
σ_4	$-e_2 = (0, -1)$
σ_{12}	$e_1 = (1, 0), e_2 = (0, 1)$
σ_{23}	$e_2 = (0, 1), -e_1 = (-1, 0)$
σ_{34}	$-e_1 = (-1, 0), -e_2 = (0, -1)$
σ_{14}	$e_1 = (1, 0), -e_2 = (0, -1)$

Figure 1.18: Fan Σ

- The dual cone σ_{12}^* of σ_{12} is generated by $\{e_1^*, (e_2)^*\}$ which are also the generators of the monoid $S_{\sigma_{12}}$.

  By the map $\phi : \mathbb{Z}^2 \to \mathbb{C}[z, z^{-1}]$,

  $$\phi(1, 0) = z_1 \ \text{ and } \ \phi(0, 1) = z_2.$$

  Since there is no relation between the monomials z_1 and z_2,

  $$R_{\sigma_{12}} = \mathbb{C}[z_1, z_2] = \mathbb{C}[x_1, x_2] \ \text{ and } \ X_{\sigma_{12}} = \mathbb{C}^2.$$

- The dual cone $\sigma_{23}^* = Pos(e_2^*, (-e_1)^*)$, and the monoid $S_{\sigma_{23}}$ is generated by $\{e_2^*, (-e_1)^*\}$.
  By the map $\phi : \mathbb{Z}^2 \to \mathbb{C}[z, z^{-1}]$,

  $$\phi(0, 1) = z_2 \ \text{ and } \ \phi(-1, 0) = z_1^{-1}.$$

  Monomials z_2 and z_1^{-1} have no relation. Thus,

  $$R_{\sigma_{23}} = \mathbb{C}[z_2, z_1^{-1}] = \mathbb{C}[x_1, x_2] \ \text{ and } \ X_{\sigma_{23}} = \mathbb{C}^2.$$

- $\sigma_{34}^* = Pos((-e_1)^*, (-e_2)^*)$, and the monoid $S_{\sigma_{34}}$ has the same generators as σ_{34}^*.
  By the map $\phi : \mathbb{Z}^2 \to \mathbb{C}[z, z^{-1}]$,

  $$\phi(-1, 0) = z_1^{-1} \ \text{ and } \ \phi(0, -1) = z_2^{-1}.$$

  Since the monomials z_1^{-1} and z_2^{-1} have no relation,

  $$R_{\sigma_{34}} = \mathbb{C}[z_1^{-1}, z_2^{-1}] = \mathbb{C}[x_1, x_2] \ \text{ and } \ X_{\sigma_{34}} = \mathbb{C}^2.$$

- The dual cone σ_{14}^* is generated by $\{e_1^*, (-e_2)^*\}$ and $S_{\sigma_{14}}$ has the same generators.
  By the map $\phi : \mathbb{Z}^2 \to \mathbb{C}[z, z^{-1}]$,

  $$\phi(1, 0) = z_1 \ \text{ and } \ \phi(0, -1) = z_2^{-1}.$$

  The monomials z_1 and z_2^{-1} has no relation,

  $$R_{\sigma_{14}} = \mathbb{C}[z_1, z_2^{-1}] = \mathbb{C}[x_1, x_2] \ \text{ and } \ X_{\sigma_{14}} = \mathbb{C}^2.$$

Now, we have the toric variety for each maximal cone of the fan Σ. In order to obtain the toric variety of the fan, we also need to consider the common faces of the cones as the gluing map will be define on them.

The cone σ_{12} has a common face σ_1 with the cone σ_{14}. We have the following gluing map between $X_{\sigma_{12}} = Specmax(\mathbb{C}[z_1, z_2])$ and $X_{\sigma_{14}} = Specmax(\mathbb{C}[z_1, z_2^{-1}])$ along σ_1 :

$$\phi_{12}^{14} : X_{\sigma_{12}} \setminus (x_2 = 0) \to X_{\sigma_{14}} \setminus (x_2^{-1} = 0) \text{ with } \phi_{12}^{14}((x_1, x_2)) = (x_1, x_2^{-1}).$$

The cone σ_{12} has a common face σ_2 with the cone σ_{23}. We have the following gluing map between $X_{\sigma_{12}} = Specmax(\mathbb{C}[z_1, z_2])$ and $X_{\sigma_{23}} = Specmax(\mathbb{C}[z_1^{-1}, z_2])$ along σ_2 :

$$\phi_{12}^{23} : X_{\sigma_{12}} \setminus (x_1 = 0) \to X_{\sigma_{23}} \setminus (x_1^{-1} = 0) \text{ with } \phi_{12}^{23}((x_1, x_2)) = (x_1^{-1}, x_2).$$

The cone σ_{23} has a common face σ_3 with the cone σ_{34}. We have the following gluing map between $X_{\sigma_{23}} = Specmax(\mathbb{C}[z_1^{-1}, z_2])$ and $X_{\sigma_{34}} = Specmax(\mathbb{C}[z_1^{-1}, z_2^{-1}])$ along σ_3 :

$$\phi_{23}^{34} : X_{\sigma_{23}} \setminus (x_2 = 0) \to X_{\sigma_{34}} \setminus (x_2^{-1} = 0) \text{ with } \phi_{23}^{34}((x_1^{-1}, x_2)) = (x_1^{-1}, x_2^{-1}).$$

The cone σ_{34} has a common face σ_4 with the cone σ_{14}. We have the following gluing map between $X_{\sigma_{34}} = Specmax(\mathbb{C}[z_1^{-1}, z_2^{-1}])$ and $X_{\sigma_{14}} = Specmax(\mathbb{C}[z_1, z_2^{-1}])$ along σ_4 :

$$\phi_{14}^{34} : X_{\sigma_{14}} \setminus (x_1 = 0) \to X_{\sigma_{34}} \setminus (x_1^{-1} = 0) \text{ with } \phi_{34}^{14}((x_1, x_2^{-1})) = (x_1^{-1}, x_2^{-1}).$$

The variety X_Σ of the fan Σ consists of the varieties $X_{\sigma_{ij}}$ and the date of the gluing maps.

Now, consider coordinates $[x_0 : x_1]$ of $\mathbb{P}^1$ and $[y_0 : y_1]$ of another $\mathbb{P}^1$ and the map $\alpha : X_{ij} \to \mathbb{P}^1 \times \mathbb{P}^1$ sending (z_1, z_2) to $(\frac{x_0}{x_1}, \frac{y_0}{y_1})$, where $X_{\sigma_{ij}} \in \Sigma$.

Please note that the map α is onto as

$$\alpha(X_{\sigma_{12}}) = \alpha(z_1, z_2) = (\frac{x_0}{x_1}, \frac{y_0}{y_1}) \text{ where } x_1, y_1 \neq 0$$

$$\alpha(X_{\sigma_{23}}) = \alpha(z_1^{-1}, z_2) = (\frac{x_1}{x_0}, \frac{y_0}{y_1}) \text{ where } x_0, y_1 \neq 0$$

$$\alpha(X_{\sigma_{34}}) = \alpha(z_1^{-1}, z_2^{-1}) = (\frac{x_1}{x_0}, \frac{y_1}{y_0}) \text{ where } x_0, y_0 \neq 0$$

$$\alpha(X_{\sigma_{14}}) = \alpha(z_1, z_2^{-1}) = (\frac{x_0}{x_1}, \frac{y_1}{y_0}) \text{ where } x_1, y_0 \neq 0$$

moreover, it can be checked that it is an isomorphism so that $X_\Sigma \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 1.2.1.4 (Cox, Little, Schenk [4]). *Let $\Sigma_1 \in \mathbb{R}^n$ and $\Sigma_2 \in \mathbb{R}^m$ be two fans. Then,*

$$\Sigma_1 \times \Sigma_2 = \{\sigma_1 \times \sigma_2 : \sigma_i \in \Sigma_i\} \ \text{ is a fan in } \mathbb{R}^n \times \mathbb{R}^m$$

$$\text{and } X_{\Sigma_1 \times \Sigma_2} = X_{\Sigma_1} \times X_{\Sigma_2}.$$

Example 1.2.1.5. *Consider the fan $\Sigma_1 \in \mathbb{R}$ consisting of the cones*

$$\sigma_{11} = Pos(e_1) \text{ and } \sigma_{12} = Pos(-e_1), \text{illustrated in Figure } 1.19,$$

and the fan $\Sigma_2 \in \mathbb{R}$ consisting of the cones $\sigma_{21} = Pos(e_2)$ and $\sigma_{22} = Pos(-e_2)$ demonstrated in Figure 1.20.
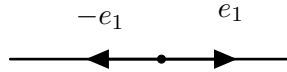
Figure 1.19: Fan Σ_1
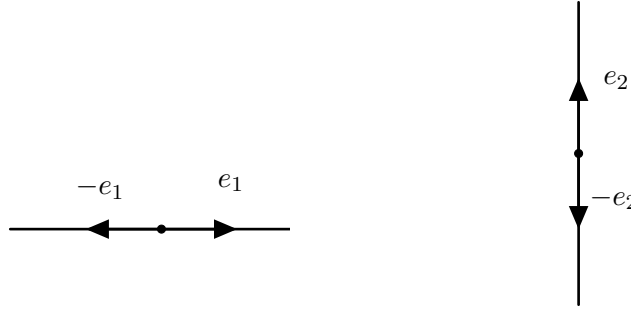
Figure 1.20: Fan Σ_2

The corresponding varieties are $X_{\Sigma_1} = \mathbb{P}^1$ and $X_{\Sigma_2} = \mathbb{P}^1$.
The fan $\Sigma_1 \times \Sigma_2$ consist of the fallowing cones as illustrated in Figure 1.18

Cones	Generators
σ_1	$e_1 = (1,0)$
σ_2	$e_2 = (0,1)$
σ_3	$-e_1 = (-1,0)$
σ_4	$-e_2 = (0,-1)$
σ_{12}	$e_1 = (1,0), e_2 = (0,1)$
σ_{23}	$e_2 = (0,1), -e_1 = (-1,0)$
σ_{34}	$-e_1 = (-1,0), -e_2 = (0,-1)$
σ_{14}	$e_1 = (1,0), -e_2 = (0,-1)$

and the variety is $X_{\Sigma_1 \times \Sigma_2} = X_{\Sigma_1} \times X_{\Sigma_2} = \mathbb{P}^1 \times \mathbb{P}^1$ as given in the Example 1.2.1.3.

In the previous sections, for a given polytope, we defined the normal fan of the polytope. Now we also now that as soon as we have a fan, we can compute the toric variety of it. In other words, when we are given a polytope $P \in \mathbb{R}^d$, we are given the normal fan $N(P)$, i.e., it is a complete fan, we also have $X_{N(P)}$.

Example 1.2.1.6. *The Hirzebruch surface $\mathcal{H}_n$ is the toric variety of the fan demonstrated in Figure 1.21 consisting of the following cones:*
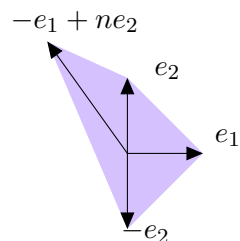
Cones	Generators
σ_1	e_1
σ_2	e_2
σ_3	$-e_2$
σ_4	$-e_1 + ne_2$
σ_{12}	e_1, e_2
σ_{13}	$e_1, -e_2$
σ_{34}	$-e_2, -e_1 + ne_2$
σ_{24}	$e_2, -e_1 + ne_2$

Figure 1.21: The fan of the Hirzebruch surface H_n

For $n = 2$, the normal fan of the polytope $P_k = \{x \in \mathbb{R}^2 : Ax \leq k\}$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ -1 & 2 \end{pmatrix} \text{ and } k = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix},$$

is the same as the fan of the Hirzebruch surface $\mathcal{H}_2$. The polytope P_k is illustrated in Figure 1.22 and $N(P_k)$ has the following cones:

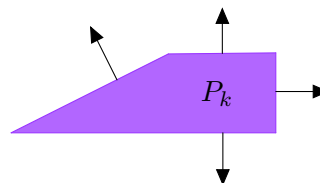| Cones	Generators
σ_1	$(1, 0)$
σ_2	$(0, 1)$
σ_3	$(0, -1)$
σ_4	$(-1, 2)$
σ_{12}	$(1, 0), (0, 1)$
σ_{13}	$(1, 0), (0, -1)$
σ_{34}	$(0, -1), (-1, 2)$
σ_{24}	$(0, 1), (-1, 2)$

Figure 1.22: The polytope P_k whose normal fan is the fan of $\mathcal{H}_2$
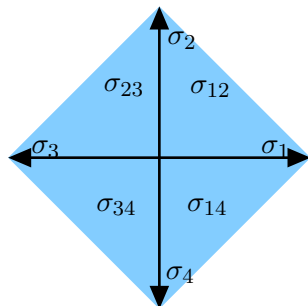
Remark 1.2.1.7. *For a d-dimensional simplex Δ_d, the toric variety of the normal fan $N(\Delta_d)$ is $X_{N(\Delta_d)} = \mathbb{P}^d$.*

Definition 1.2.1.8. *Let Σ be a fan in $\mathbb{R}^m$ and let σ be a smooth cone in Σ with generators $\{v_1, \ldots, v_m\}$. Let $v_0 = v_1 + \ldots + v_m$ and $\Sigma'(\sigma)$ be the set of all cones generated by the subsets of $\{v_0, \ldots, v_m\}$ not containing $\{v_1, \ldots, v_m\}$. Then*

$$\Sigma^* = \Sigma \setminus \{\sigma\} \cup \Sigma'(\sigma)$$

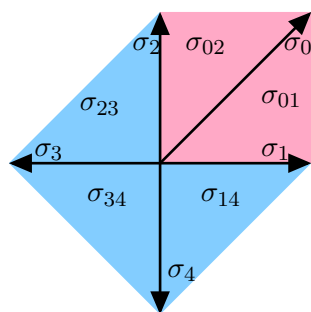*is a fan in $\mathbb{R}^m$ and called the star subdivision of Σ along σ.*

Example 1.2.1.9. *Consider the fan Σ illustrated in Figure 1.23 containing the following cones:*

Cones	Generators
σ_1	$v_1 = (1,0)$
σ_2	$v_2 = (0,1)$
σ_3	$v_3 = (-1,0)$
σ_4	$v_4 = (0,-1)$
σ_{12}	$v_1 = (1,0), v_2 = (0,1)$
σ_{23}	$v_2 = (0,1), v_3 = (-1,0)$
σ_{34}	$v_3 = (-1,0), v_4 = (0,-1)$
σ_{14}	$v_1 = (1,0), v_4 = (0,-1)$

Figure 1.23: Fan Σ

Consider the smooth cone $\sigma_{12} \in \Sigma$ and let $v_0 = (1,0) + (0,1) = (1,1)$. Then we have $\Sigma'(\sigma_{12}) = \{\sigma_0 = Pos(v_0), \sigma_{01} = Pos(v_0, v_1), \sigma_{02} = Pos(v_0, v_2)\}$ as the set of all cones generated by subsets of $\{v_0, \ldots, v_4\}$ that does not contain $\{v_1, \ldots, v_4\}$. By the definition, the star subdivision of Σ along σ_{12} is $\Sigma^* = \Sigma \setminus \{\sigma_{12}\} \cup \Sigma'$ which is the fan illustrated in Figure 1.24 with the following cones:

Cones	Generators
σ_0	$v_0 = (1,1)$
σ_1	$v_1 = (1,0)$
σ_2	$v_2 = (0,1)$
σ_3	$v_3 = (-1,0)$
σ_4	$v_4 = (0,-1)$
σ_{01}	$v_0 = (1,1), v_1 = (1,0)$
σ_{02}	$v_0 = (1,1), v_2 = (0,1)$
σ_{23}	$v_2 = (0,1), v_3 = (-1,0)$
σ_{34}	$v_3 = (-1,0), v_4 = (0,-1)$
σ_{14}	$v_1 = (1,0), v_4 = (0,-1)$

Figure 1.24: Fan Σ^*

Definition 1.2.1.10. *Given a fan Σ, we say that a fan Σ' refines Σ if every cone in Σ' is contained in a cone of Σ and $|\Sigma| = |\Sigma'|$ (the support of a fan Σ is $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$).*

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Proposition 1.2.1.11 (Cox, Little, Schenk [4])**.** Σ^* *is a refinement of* Σ *and the induced toric morphism* $\phi : X_{\Sigma^*} \to X_{\Sigma}$ *makes* X_{Σ^*} *the blow up of* X_{Σ} *at the distinguished point* γ_σ *corresponding to the cone* σ*.*

Chapter 2

Chamber Complex

2.1 Chamber

Given a polytope $P \subset \mathbb{R}^d$, the natural question arises of whether P is decomposable, and if it is, whether we can know the summands. In 1973, McMullen[10] presented a method to answer this question by defining the type cone (Minkowski cone or chamber) of a polytope. This theory started to have a quite important role in the theory of vector partition functions by the paper of Sturmfels [11] proving that for each chamber, there is an associated piece-wise polynomial. Further studies were made on the chambers and vector partition functions by Brion et al.[3], Beck [1] and others. Moreover, Henk et al. in [8] considered integer decomposition of polytopes and their application to integer programming. Emiris et al.[6] and Brion et al.[3] also provided different algorithms to compute the chamber of a polytope.

In this section, we give the definition of the chamber of a given polytope and provide existed methods to compute it.

In 1973, McMullen[10] defined the type cone (chamber, Minkowski cone) of a polytope as follows.

Definition 2.1.0.1. *For a given polytope $P = \left\{ x \in \mathbb{R}^d : Ax \leq b \right\}$ for some matrix $A \in \mathbb{R}^{m \times d}$ and some vector $b \in \mathbb{R}^m$, the chamber (type cone, Minkowski cone) C_{P_b} of P_b is the cone of vectors $c \in \mathbb{R}^m$ for which P_c is strongly combinatorially equivalent to P_b.*

Moreover, he proved the following:

Theorem 2.1.0.2 (McMullen [10]). *A d-dimensional polytope P is indecomposable if and only if the chamber C_P is one dimensional.*

Theorem 2.1.0.3 (McMullen [10]). *For a given d-dimensional polytope P, the chamber C_P is a polyhedral complex.*

Theorem 2.1.0.4 (McMullen [10]). *For a given d-dimensional polytope P, each face of the closure of the chamber C_P is a chamber.*

Theorem 2.1.0.2 gives the answer of the question asking whether a given polytope P_b is decomposable as by the theorem, it is enough to check the dimension of C_{P_b}. If C_{P_b} has dimension more than one, then we can find some information about the indecomposable summands of P_b by considering the generating rays of C_P. By Theorem 2.1.0.4, the faces

of the chamber are also chambers. Thus, while points c taken from the interior of C_P give polytopes that are strongly combinatorially equivalent to P, polytopes P_d for points taken from one of the generating rays of C_{P_b} are strongly combinatorial equivalent to one of the indecomposable summands of P_b.

We provide examples and more details about the chamber of a given polytope in the following parts of this section. Let us explain the process of computing the chamber of a polytope considering the method given by Brion et al. [3].

To begin with, we fist need to define the cone of non-empty polytopes of a given matrix $A \in \mathbb{R}^{m \times d}$.

Definition 2.1.0.5. *For a given matrix $A \in \mathbb{R}^{m \times d}$ of rank d, the set $C(A)$ of non-empty polytopes is the set of vectors $b \in \mathbb{R}^m$ for which*

$$P_b = \left\{ x \in \mathbb{R}^d : Ax \leq b \right\} = \left\{ x \in \mathbb{R}^d : [A, I] \begin{bmatrix} x \\ y \end{bmatrix} = b, \; y \geq 0 \right\}$$

is non-empty, i.e.,
$$C(A) = \{ b \in \mathbb{R}^m : P_b \neq \emptyset \}.$$

Since polytopes P_b are non-empty if and only if the system $Ax \leq b$ has a non-empty solution set, and one can add slack variables to the system of inequality to get a system of equalities,
$[A, I] \begin{bmatrix} x \\ y \end{bmatrix} = b$, where I is the identity matrix and $y \in \mathbb{R}^m_{\geq 0}$ so that,

$$C(A) = \{ b \in \mathbb{R}^m : P_b \neq \emptyset \} = C(A, I) = \{ b \in \mathbb{R}^m : P_b \neq \emptyset \}.$$

The system $[A, I] \begin{bmatrix} x \\ y \end{bmatrix} = b$ of equalities has a non-empty solution if and only if the vector b is a linear combination of the columns of the matrix A and positive linear combinations of the columns of I. Thus,

$$C(A) = C(A, I) = \left\{ b \in \mathbb{R}^m : b = \sum_{i=1}^{d} \lambda_i a_i + \sum_{j=1}^{m} \beta_j e_j : \begin{array}{l} a_i \text{ is a column of A} \\ e_j \text{ is a column of I} \\ \lambda_i \in \mathbb{R}, \beta_j \in \mathbb{R}_{\geq 0} \end{array} \right\}.$$

For each b of the form $b = \sum_{i=1}^{d} \lambda_i a_i + \sum_{j=1}^{m} \beta_j e_j$, $(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ is a solution for the system of equalities.

In other words, the set $C(A) = C(A, I)$ is a cone whose generating lines are the columns of A, and whose generating rays are the columns of the identity matrix.

As mentioned above, we may get polytopes P_b of different combinatorial types by inserting different vectors b which corresponds to moving the halfspaces. The cone $C(A)$ contains all such vectors. Moreover, for a vector b taken from $C(A)$, P_b is non-empty and since the translations $P_b + t = P_{b+At}$ are also non-empty, $b + At \in C(A)$.

As the polytopes that are the translations of P_b are homothetic to P_b, we do not want to distinguish between vectors that give rise to translations of P_b in $C(A)$. McMullen

[10], Brion et al. [3] and Henk et al. [8] consider representations of polytopes such that for each polytope and its translations, there is only one representative.

For any $b \in C(A)$, and any vector $t \in \mathbb{R}^d$,

$$P_b + t = \left\{ x + t \in \mathbb{R}^d : Ax \leq b \right\} = \left\{ y \in \mathbb{R}^d : Ay \leq At + b \right\} = P_{b+At}.$$

We define an equivalence relation $\sim$ on $C(A)$ as

$$b \sim b' \iff b - b' \in A\mathbb{R}^d.$$

Lemma 2.1.0.6. *The relation $\sim$ defined above is an equivalence relation on $C(A)$.*

Proof.   • $\sim$ is reflexive, since for all $b \in C(A)$,

$$b - b = 0 \in A\mathbb{R}^d \iff b \sim b$$

   • $\sim$ is symmetric, since for all $a, b \in C(A)$,

$$b \sim a \Rightarrow b - a = c \in A\mathbb{R}^d \Rightarrow a - b = -c \in A\mathbb{R}^d \Rightarrow a \sim b.$$

   • $\sim$ is transitive, since for all a, b, c with $a \sim b$ and $b \sim c$,

$$a - b \in A\mathbb{R}^d \text{ and } b - c \in A\mathbb{R}^d \Rightarrow a - b + b - c \in A\mathbb{R}^d \Rightarrow a - c \in A\mathbb{R}^d \Rightarrow a \sim c.$$

$\square$

Now we would like to have one representative for each equivalence class.

Lemma 2.1.0.7. $C(A) \bigcap \left\{ b \in \mathbb{R}^m : A^T b = 0 \right\}$ *has one representative from each equivalence class.*

Proof. Assume that $b, b' \in C(A) \bigcap \left\{ b \in \mathbb{R}^m : A^T b = 0 \right\}$ with $b \sim b'$ and $b \neq b'$. Then we have that $b - b' \in A\mathbb{R}^d$, but $b - b'$ is also in $\left\{ b \in \mathbb{R}^m : A^T b = 0 \right\}$, since they are taken from the intersection. We also have that $\left\{ b \in \mathbb{R}^m : A^T b = 0 \right\}$ is orthogonal to $A\mathbb{R}^d$. Since for two orthogonal spaces $S, S^\perp$, we know that $S \cap S^\perp = \vec{0}$, $b - b' = 0$ so that $b = b'$. This is a contradiction. $\square$

The lines of the cone $C(A)$ stands for translations, and when we intersect it with the orthogonal space that is the right kernel of A^T, the intersection has only one point such that it has one representative from each equivalence class.

Thus the cone of non-empty polytopes that does not distinguish between translations is

$$\tilde{C}(A) = \left\{ b \in \mathbb{R}^m : P_b \neq \emptyset \text{ and } A^T b = 0 \right\} = C(A) \bigcap \left\{ b \in \mathbb{R}^m : A^T b = 0 \right\}.$$

The chamber (or type cone, or Minkowski cone) of a polytope P_b is the cone of all vectors b, that give polytopes that are strongly combinatorially equivalent to P_b so that the type cone of P_b is indeed a subdivision of $\tilde{C}(A)$. In other words, among all the vectors

$c \in \tilde{C}(A)$, we are interested in ones that give polytopes that are strongly combinatorially equivalent to the given polytope P_b.

As we have a fixed set of outer normal vectors given by A, the set of normal vectors of defining half spaces of non-empty polytopes that we can obtain, are subsets of A, so that outer normals do not change, but some half spaces might become redundant. When there are redundant half spaces, the combinatorial type changes. It means that there might be some vectors $c \in \tilde{C}(A)$ that give polytopes of different combinatorial types.

For $d-$dimensional simple polytopes, a vertex is the intersection of d hyperplanes. Having redundant half spaces means that a vertex, defined by the intersection of the redundant half spaces, is an element of a face of the polytope or it is not a part of the polytope anymore.

Example 2.1.0.8. *For $A = \{a_1 = (1,0), a_2 = (0,1), a_3 = (-1,0), a_4 = (0,-1), a_5 = (1,1)\}$, the change of the vertices, so that the combinatorial type of polytopes*

$$P_a = \left\{x \in \mathbb{R}^5 : Ax \leq a\right\}, P_b = \left\{x \in \mathbb{R}^5 : Ax \leq b\right\}$$

for vectors $a = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 3 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$ respectively, is illustrated below.
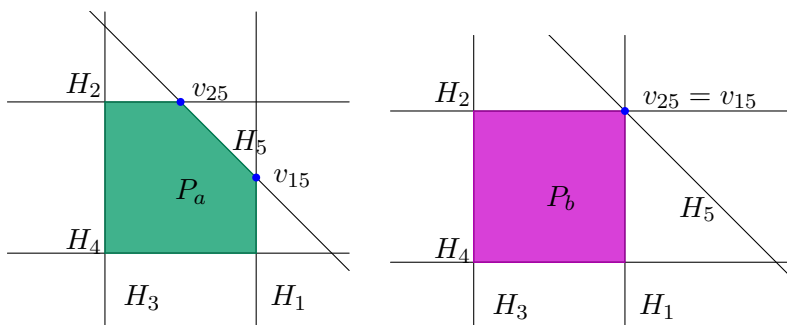
When we move the hyperplane H_5 by replacing the vector a with the vector b, two vertices v_{25} and v_{15} become the same, so that the relation between faces changes.

Since the vertices are important for detecting the combinatorial type, we use them to obtain the chamber of a given polytope as follows.

Let us take d subsets S of $[m]$. For each S and for each s in S, we shall turn $H_s = \left\{x \in \mathbb{R}^d : \langle a_s, x \rangle \leq b_s\right\}$ into $\hat{H}_s = \left\{x \in \mathbb{R}^d : \langle a_s, x \rangle = b_s\right\}$ for some $b_s \in \mathbb{R}$. We are interested in the vectors b, for which

$$P_b = \bigcap_{i \in [m] \backslash S} H_i \bigcap_{j \in S} \hat{H}_j$$

contains at least the vertex v_S.

Going back to the expression with slack variables, this process corresponds to adding slack variables to the rows a_i for $i \in [m] \setminus S$ and not adding slack variables for the rows a_j for $j \in S$. Thus the cone $C(A, I_S)$ where $I_S = \{e_i \in I : i \in [m] \setminus S\}$ is the cone that contains all vectors b, for which

$$P_b = \left\{ x \in \mathbb{R}^d : [A, I_S] \begin{bmatrix} x \\ y \end{bmatrix} = b,\, y \geq 0 \right\} = \bigcap_{i \in [m] \setminus S} H_i \bigcap_{j \in S} \hat{H}_j.$$

is non-empty.

For any $b \in C(A, I)$, the chamber (or type cone, or Minkowski cone) of P_b is the intersection

$$C_{P_b} = \left(\bigcap_{S \subset [m], |S| = d,\, b \in C(A, I_S)} C(A, I_S) \right) \bigcap \left\{ b \in \mathbb{R}^m : A^T b = 0 \right\}.$$

When we do not take intersection with the right kernel of A^T, the chamber contains lines generated by the columns of the matrix A, which stand for the translations. Since we do not want to distinguish between translations, we have the equivalence relation we defined above and we take the intersection with $\left\{ b \in \mathbb{R}^m : A^T b = 0 \right\}$ to have one representative from each equivalence class.

The dimension of the chamber of a polytope P_b gives information about its decomposability. If the chamber C_{P_b} has dimension bigger than 1, P_b is decomposable. Moreover, points c taken from the generating rays of C_{P_b} give polytopes that are of the same combinatorial type as indecomposable summands P_c of P_b, and points d taken from the faces of C_{P_b} give the polytopes P_d that are of the same combinatorial type as decomposable summands of P_b.

Example 2.1.0.9. *For a given matrix* $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$,

The cone of non-empty polytopes of A is the following

$$\tilde{C}(A) = \tilde{C}(A, I) = C(A, I) \bigcap \left\{ b \in \mathbb{R}^5 : A^T b = 0 \right\}.$$

$C(A, I)$: *A* 5 *dimensional polyhedron in* $\mathbb{R}^5$ *defined as the convex hull of one vertex, 3 rays and 2 lines.*

$$\text{Rays: } [[-1, 0, 1, 0, 0], [1, 0, 0, 0, 0], [0, 1, 0, 0, 0]]$$

$$\text{Lines: } [[1, 0, -1, 0, 1], [1, -1, -1, 1, 0]]$$

$\left\{ b \in \mathbb{R}^5 : A^T b = 0 \right\}$: 3 *dimensional polyhedron in* $\mathbb{R}^5$ *defined as convex hull of one vertex and 3 lines.*

$$\text{Rays: } [[1, 1, 0, 0, -1], [0, 1, 0, 1, 0], [1, 0, 1, 0, 0]]$$

$\tilde{C}(A):$ *A 3 dimensional polyhedron in* $\mathbb{R}^5$ *defined as the convex hull of one vertex and 3 rays.*

Rays: $[[5, 1, 3, -1, -2], [1, 5, -1, 3, -2], [-1, -1, 1, 1, 2]]$

For each 2−subsets of $[5]$*, the cone* $C(A, I_S)$ *is illustrated in the table below.*

Table 2.1: Subsets S, the hyperplanes they fix and their cone of non-empty polytopes

S	Fixed Hyperplanes	Rays of $C(A, I_S)$
$\{1, 2, 3\}$		$[0, 0, 1, 0, 0], [0, 1, 0, 0, 0],$ $[1, 0, 0, 0, 0]$
$\{1, 2, 4\}$		$[-1, 1, 1, 0, 0], [0, 1, 0, 0, 0],$ $[1, 0, 0, 0, 0]$
$\{1, 2, 5\}$		$[-1, 0, 1, 0, 0], [0, 1, 0, 0, 0],$ $[1, 0, 0, 0, 0]$
$\{1, 3, 4\}$		$[-1, 1, 1, 0, 0], [0, 0, 1, 0, 0],$ $[1, 0, 0, 0, 0]$
$\{1, 3, 5\}$		$[-1, 0, 1, 0, 0], [1, 0, 0, 0, 0]$

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$\{1,4,5\}$		$[-1,0,1,0,0],[-1,1,1,0,0],$ $[1,0,0,0,0]$
$\{2,3,4\}$		$[-1,1,1,0,0],[0,0,1,0,0],$ $[0,1,0,0,0]$
$\{2,3,5\}$		$[-1,0,1,0,0],[0,0,1,0,0],$ $[0,1,0,0,0]$
$\{2,4,5\}$		$[-1,0,1,0,0],[0,1,0,0,0]$
$\{3,4,5\}$		$[-1,0,1,0,0],[-1,1,1,0,0],$ $[0,0,1,0,0]$

All cones listed above have the following lines:

$$Lines: [1,0,-1,0,1], [1,-1,-1,1,0].$$

For $b=(2,2,0,0,3)$, the chamber C_{P_b} is the intersection

$$\left(\bigcap_{S \subset [m], |S|=d, b \in C(A,I_S)} C(A,I_S) \right) \bigcap \{b \in \mathbb{R}^m : A^T b = 0\}$$

which is the 3-dimensional polyhedron with the following rays:

Rays: $[1,1,1,1,0]\,,[3,-1,5,1,2]\,,[-1,3,1,5,2]$
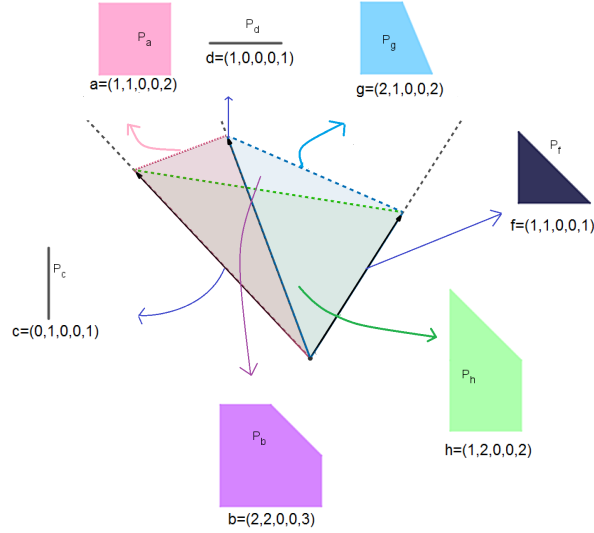
Figure 2.1: The chamber of P_b

Consider the closure of the chamber C_{P_b}. The generating rays of C_{P_b} are the chambers of the indecomposable summands of P_b so that for any point c taken form one of the generating rays of C_{P_b}, P_c is an indecomposable summand of P_b. In Figure 2.1, we see that points from the three generating rays r_1, r_2, r_3 of C_{P_b} give a triangle, a vertical line segment and a horizontal line segment. The right hand side vector v of the Minkowski sum of those three polytopes is in the interior of the chamber C_{P_b} and P_v is of the same combinatorial type as P_b.

  When we consider the faces of the closure of C_{P_b}, we see that for a point g taken from a face of C_{P_b} gives the polytope P_g which is a decomposable summand of P_b. P_g is decomposable, as its chamber which is the face of C_{P_b} is two dimensional and generated by two rays which give information about P_g's indecomposable summands. We can see in Figure that we get squares when we take a point from the face of C_{P_b}, which is generated by rays that give line segments, and a square is the Minkowski sum of two line segments.

Example 2.1.0.10. *For a given matrix* $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$,

$$\tilde{C}(A) = \tilde{C}(A, I) = C(A, I) \bigcap \left\{ b \in \mathbb{R}^4 : A^T b = 0 \right\}.$$

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$C(A, I)$: *A 4 dimensional polyhedron in $\mathbb{R}^4$ defined as the convex hull of one vertex, 2 rays and 2 lines.*

$$\text{Rays: } [[2, -1, 0, 0], [-1, 1, 0, 0]]$$

$$\text{Lines: } [[1, 0, -1, 0, 1], [1, -1, -1, 1, 0]]$$

$\left\{b \in \mathbb{R}^4 : A^T b = 0\right\}$: *A 2 dimensional polyhedron in $\mathbb{R}^4$ defined as convex hull of one vertex and 2 lines.*

$$\text{Lines: } [[1, 1, 0, 1], [1, 2, 1, 0]]$$

$\tilde{C}(A)$: *A 2 dimensional polyhedron in $\mathbb{R}^4$ defined as the convex hull of one vertex and 2 rays.*

$$\text{Rays: } [[0, 1, 1, -1], [1, 0, -1, 2]]$$

For $b = (0, 0, 7, 4)$, the chamber C_{P_b} illustrated in figure 2.2 is the intersection

$$\left(\bigcap_{S \subset [m], |S| = d, b \in C(A, I_S)} C(A, I_S) \right) \bigcap \left\{ b \in \mathbb{R}^m : A^T b = 0 \right\}$$

which is the 2-dimensional polyhedron in $\mathbb{R}^4$ with the following rays:

$$\text{Rays: } [[1, 1, 0, 1], [1, 2, 1, 0]]$$
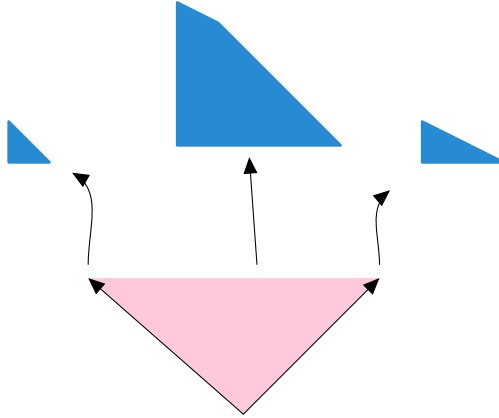
Figure 2.2: Chamber of the Minkowski sum of two triangles

Here, we observe that the polytope P_b is the Minkowski sum of two triangles such that points taken from the generating rays of C_{P_b} give polytopes that are strongly combinatorially equivalent to the indecomposable summands of P_b.

Now, let us consider the chamber C_{P_d} for $d = (1, 1, 0, 1)$.

It is a 1 dimensional polyhedron in $\mathbb{R}^4$ generated by the ray $[1, 1, 0, 1]$. It is indeed the generating ray of the chamber C_{P_b}.

2.2 Chamber Complex

In the previous sections we defined the chamber of a polytope $P_b = \left\{ x \in \mathbb{R}^d : Ax \leq b \right\}$ for some $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$ that is the cone of vectors c for which the polytopes P_c are strongly combinatorial equivalent to P_b. When we have a fixed matrix $A \in \mathbb{R}^{m \times d}$ and we change the vector b, we can have an empty polytope or polytopes that are strongly combinatorial equivalent or polytopes of different combinatorial types. The reason we can have those cases is that while moving the half spaces by changing right hand side vectors, the intersection might change.

We are interested in finding all combinatorial types one can obtain for a fixed matrix A. In other words, we would like to find all different chambers we can have such that each chamber is the set of vectors c for which polytopes P_c are strongly combinatorial equivalent.

In this section we define the chamber complex for a given matrix A, present an algorithm to compute it inspired by previous works by [10], [11], [3] and others.

Definition 2.2.0.1. *Given a matrix $A \in \mathbb{R}^{m \times d}$ of rank d, the chamber complex of A is the collection of all chambers that can be obtained e.i., points form the same chamber give poltopes that are strongly combinatorially equivalent when they are inserted as right hand side vectors in the system $Ax \leq b$ of inequality.*

As we mentioned at the beginning, for a given matrix $A = \{a_1, \cdots, a_m\} \in \mathbb{R}^{m \times d}$ of rank d, we have a fixed set of outer normal vectors of the half spaces H_i for some $b_i \in \mathbb{R}$ for $i \in [m]$. Changing the right hand side vectors b corresponds to moving those hyperplanes. While moving the hyperplanes, the intersection of half spaces so that the polytope we obtain, changes. When some half spaces become redundant during this process we obtain polytopes of different combinatorial types. Please note that being polytopes of different combinatorial types here means that the polytopes have different normal fans.

Example 2.2.0.2. *One can observe from the figure 2.3 that when we move the half spaces whose outer normal vectors are given by the matrix*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}$$

we may have some polytopes that are strongly combinatorially equivalent or that are of different combinatorial types.
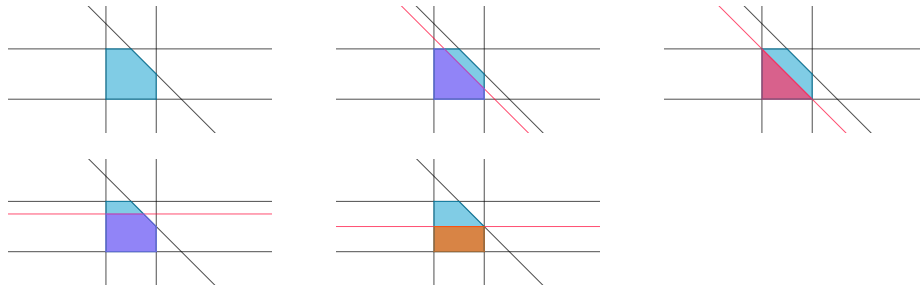
36

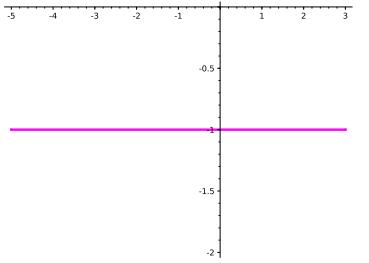Figure 2.3: Polytopes of different combinatorial types one can obtain by moving the hyperplanes

In order to illustrate the notion of the chamber complex better, let us consider the example 2.1.0.9 in terms of the chamber complex of the matrix A, instead of the chamber of a given polytope P_b for a given b.
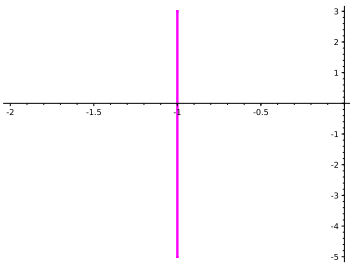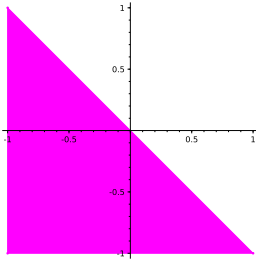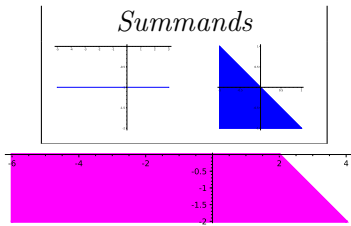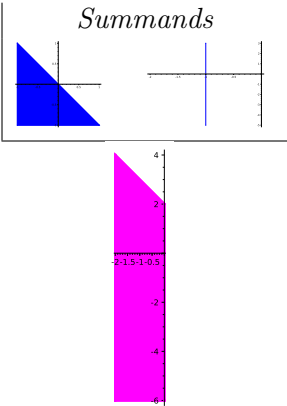
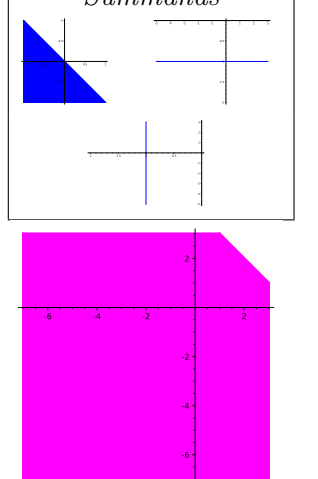**Example 2.2.0.3.** *Consider matrix*
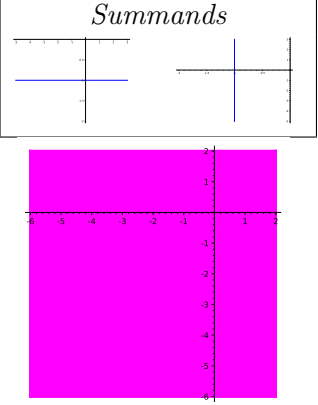
$$
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}.
$$

Chamber complex for A consists of the following chambers.

Table 2.2: List of the chambers of the chamber complex of A, and examples of the polytopes obtained by the points taken from the chambers

Chambers that the chamber complex of A has	A polytope P_b for b taken from the interior of the chamber
Chamber I: A 1-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 1 ray. **Rays:**$[3, -1, 5, 1, 2]$	

Chamber II: *A 1-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 1 ray.* **Rays:**$[-1,3,1,5,2]$	
Chamber III: *A 1-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 1 ray.* **Rays:**$[1,1,1,1,0]$	
Chamber IV: *A 2-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 2 ray.* **Rays:**$[3,-1,5,1,2]$	*Summands*
Chamber V: *A 2-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 2 ray.* **Rays:** $[1,1,1,1,0]$, $[-1,3,1,5,2]$	*Summands*

Chamber VI: *A 2-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 2 ray.* **Rays:** $[3, -1, 5, 1, 2]$, $[-1, 3, 1, 5, 2]$	*Summands*
Chamber VII: *A 3-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 3 ray.* **Rays:** $[1, 1, 1, 1, 0]$, $[3, -1, 5, 1, 2]$, $[-1, 3, 1, 5, 2]$	*Summands*

We can deduce from the table that Chamber VII is the top dimensional chamber of the chamber complex and other chambers are faces of the Chamber VII. Thus we can illustrate the chamber complex as the closure of the chamber of P_b for $b = (3, 3, 7, 7, 4)$ as in Figure 2.4 .
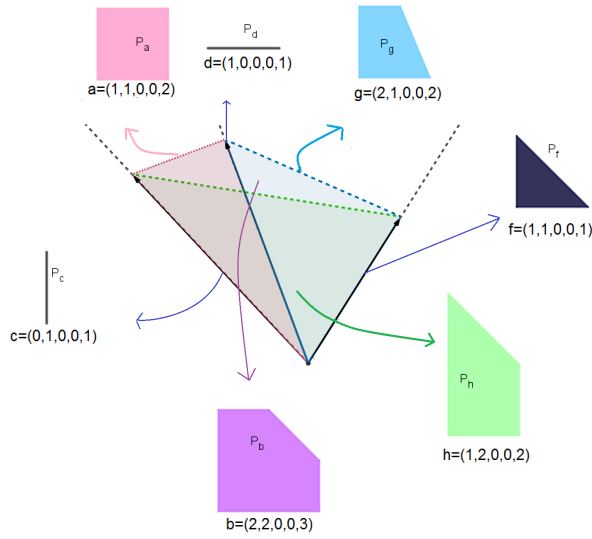
Figure 2.4: Chamber complex as a closure of a chamber

Please note that in this example there was no vector b, but only the matrix A, while in the example 2.1.0.9, we needed to be given a vector b such that we computed the chamber of P_b. Finding the chamber complex for the matrix A corresponds finding different chambers, i.e., each chamber is a set of vectors b, and stands for a different combinatorial type.

2.2.1 Number of Top Dimensional Chambers

In the example 2.2.0.3, we see that when the number of top dimensional chambers of the chamber complex of a given matrix A, we can obtain the chamber complex by considering the closure of the top dimensional chamber. Now we can ask weather a chamber complex always have only one top dimensional chamber.
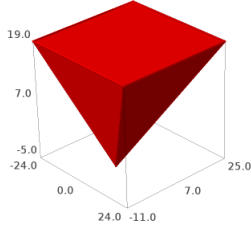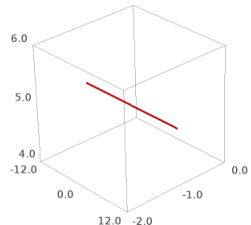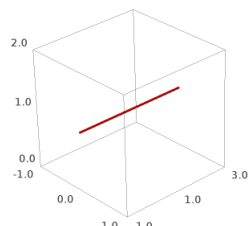
Question 2.2.1.1. *Does a matrix A whose chamber complex has more than one top dimensional chambers exist?*
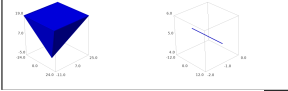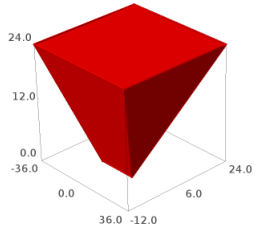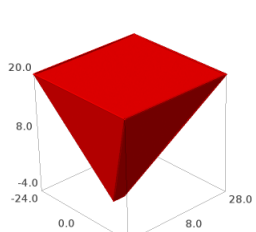
Answer 2.2.1.2. *Yes!*
For example, for the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & -2 & -1 \end{pmatrix},$$

The chamber complex of A has the following chambers:

Chambers that the chamber complex of A has	A polytope P_b for b taken from the interior of the chamber
Chamber I: A 1-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 1 ray. **Rays:**$[19, 5, 6, 5, 3]$	
Chamber II: A 1-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 1 ray. **Rays:**$[5, 7, -6, 7, -3]$	
Chamber III A 1-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 1 ray. **Rays:**$[1, -1, 2, -1, 1]$	

Chamber IV: A 2-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 2 ray. **Rays:**$[19, 5, 6, 5, 3]$, $[5, 7, -6, 7, -3]$	*Summands*
Chamber V: A 2-dimensional polyhedron in $\mathbb{Z}^5$ defined as the convex hull of 1 vertex and 2 ray. **Rays:** $[19, 5, 6, 5, 3]$, $[1, -1, 2, -1, 1]$	*Summands*

As one can observe from the given table above, the chamber IV and chamber V are top dimensional chambers we can obtain for A. Indeed we cannot express the chamber complex of A in terms of the closure of one chamber, but the union of the closure of two chambers. Namely, the union of the closure of C_{P_a} for $a = (24, 12, 0, 12, 0)$ and the closure of C_{P_b} for $b = (20, 4, 8, 4, 4)$.

Figure 2.5 demonstrates how the chamber complex of A looks like.
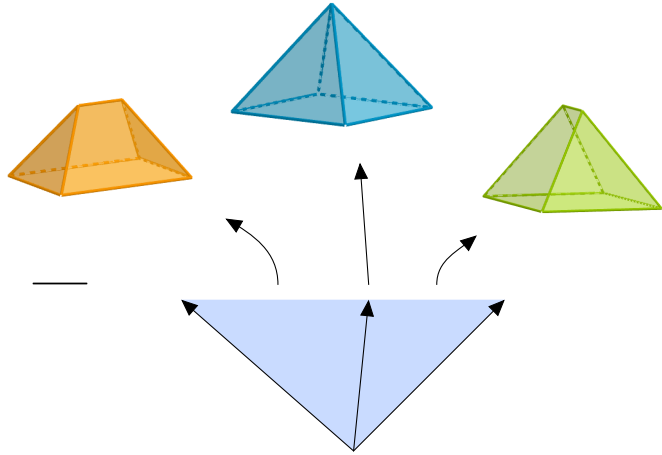
Figure 2.5: The chamber complex of A that has more than one top dimensional chambers

The chamber complex does not have the chamber C' for which, for the points v taken from C', P_v gives the Minkowski sum of two line segments P_k, P_l where k and l are taken from the chambers II and III respectively as the matrix A does not have the row $(0, 0, -1)$ which is the outer normal vector the half space that is needed to obtain a square by moving the half spaces the rows of A defines.

As in the given example 2.2.1.2, we can have cases where the chamber of a matrix A has more than one top dimensional chambers and it cannot be expressed as the closure of a chamber of a polytope. Thus for those cases, we need another method to compute the chamber complex.

2.2.2 An Algorithm to Compute the Chamber Complex

For the cases where the chamber complex of a given matrix $A \in \mathbb{R}^{m \times d}$ has more than one top dimensional chambers, let us present the following algorithm.

Data: A matrix $A \in \mathbb{R}^{m \times d}$ with $m > d$
Result: The chamber complex: the set of chambers one can get for A
if *The matrix A has full rank* **then**

    $m :=$ Number of rows of A;
    $d =:$ Number of columns of A;
    $S :=$ Subsets of $[m]$ of size d;
    $I :=$ Identity matrix of size $m \times m$;
    $Cones = \{\}$;
    for $s \in S$ **do**
        $I_s := \{e_i \in I : i \in [m] \setminus S\}$;
        $C(A, s) := Polyhedron(\text{rays} = I_s, \text{lines} = \text{columns of } A)$;
        Add $C(A, s)$ to $Cones$
    end
    $ChamberComplex := \{\}$;
    $RightKernel := Polyhedron(\text{rays=Generators of the right kernel of } A)$;
    for D *in the power set* $\mathcal{P}(Cones)$, $D \neq \emptyset$, $D \neq Cones$ **do**
        $K := \bigcap_{c \in D} c$;
        $KK := K \cap RightKernel$;
        $R :=$ Set of generating rays of K;
        if *For all* $r \in R$, $dim(P_r) > 0$ **then** Add KK to $ChamberComplex$;
    end
    for C_1, C_2 *in ChamberComplex with* $dim(C_1) = dim(C_2)$ *and* $C_1 \cap C_2 = C_1$
    do
        Remove C_2;
    end
    return *ChamberComplex*
else
    Raise Exception
end

Algorithm 2.1: Algorithm for the chamber complex

Theorem 2.2.2.1. *The algorithm 2.1 is correct.*

Proof. Please note that the restrictions on the rank of A and on the relation between number of columns and rows of A makes sure that the solution set is bounded or empty. We are given a matrix $A \in \mathbb{R}^{m \times d}$ and we would like to find all the chambers so that combinatorial types we can get by moving the half spaces given by A.

Recall that for any $b \in \mathbb{R}^m$, a polytope P_b can be expressed as follows:

$$P_b = \left\{ x \in \mathbb{R}^d : Ax \leq b \right\} = \left\{ x \in \mathbb{R}^d : [A, I] \begin{bmatrix} x \\ y \end{bmatrix} = b, \ y \geq 0 \right\}.$$

Moreover, vertex of a d dimensional simple polytope is the intersection of d hyperplanes and changes on the vertex arrangements change the combinatorial type of the polytope we obtain. We take d subsets S of $[m]$ and for each S, we add slack variables to the rows a_i for $i \in [m] \setminus S$ and do not add slack variables for the rows a_j for $j \in S$. This corresponds to fixing d hyperplanes which makes sure that points taken from the cone give polytopes whose normal fan contains the normal cone of the intersection of the d hyperplanes.

Thus for each S, the cone $C(A, I_S)$ where $I_S = \{e_i \in I : i \in [m] \setminus S\}$ is the cone that contains all vectors b, for which

$$P_b = \left\{ x \in \mathbb{R}^d : [A, I_S] \begin{bmatrix} x \\ y \end{bmatrix} = b,\, y \geq 0 \right\} = \bigcap_{i \in [m] \setminus S} H_i \bigcap_{j \in S} \hat{H}_j.$$

is non-empty.

With this process, for each possible vertex obtained by fixing d of hyperplanes, we obtain a cone of vectors b that encodes different half space arrangements for those that are not fixed. As we see in the previous section, this method was also used by [10], [3], [8].

Let

$$Cones = \{C(A, I_S) : S \subset [m],\, |S| = d,\, I_S = \{e_i \in I : i \in [m] \setminus S\}\}$$

be the set of all cones we obtain by fixing d hyperplanes. Now consider non-empty, proper subsets of $Cones$. Having those subsets corresponds to have sets of some cones that make sure of existence of the intersection of some hyperplanes.

For each non-empty proper subset D of $Cones$, the intersection $\bigcap_{c \in D} c$ is the set of points b for which the polytope P_b has the vertices that $c \in D$ defines.

As in the section 2.1, we do not want to distinguish between the translations of polytopes, for each non-empty proper subset D of $Cones$, we consider the intersection

$$\left(\bigcap_{c \in D} c \right) \bigcap \{b \in \mathbb{R}^m : A^T b = 0\}$$

However, each intersection is not a chamber because, some of them might be contained in another or some of them might consist of points that only gives zero dimensional polyopes.

Here we consider the following fact coming from the definition of type cone by McMullen in [10] to eliminate the cones that are chambers.

For points b taken from the generating rays of a type cone, the polytope P_b has dimension more than 0.

The last step of the algorithm makes sure that the cones of the same dimensional are not contained in each other as by definition, chambers are the maximal cones for which every point taken from it gives a polytope having the same normal fan when we insert the point as a right-hand-side vector. For different candidates of chambers C_1 and C_2 of the same dimension, if C_2 contains C_1, it means that C_2 is not a chamber as each chamber stands for a different combinatorial type and C_2 cannot stand for a different combinatorial type than C_1 while containing C_1.

Assume that C_2 is a chamber and C_1 is not a chamber. Since their dimensions are the same and they are different, C_2 has at least one more generator than C_1. We know that all points taken from the interior of a chamber give the polytopes of that are strongly combinatorially equivalent, and that have the Minkowski summands of the same number. Thus C_1 has the same generators as C_2 as it is contained in C_2, and C_2 is a chamber. It is a contradiction.

Please note that it would not be the case if the cones of different dimension were containing each other as the faces of the closure of a chamber are also chambers.

$\square$

The following example illustrates for each d subset of $[m]$, fixing d hyperplanes and intersecting cones of non-empty polytopes.

Example 2.2.2.2. *Consider the matrix $A = [[1,0],[0,1],[-1,0],[0,-1],[1,1]]$. When we add slack variables for all row but the second and third rows such that we have the following matrix $[A, I_{145}]$*

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

points form the cone of non-empty polytopes $C(A, I_{145})$ have points b such that the solution set of $[A, I_{145}][x, y] \leq b$ is the vertex $v_{2,3}$. Some of such vectors b can give different supporting hyperplane arrangements of halfspaces that give polytopes of different combinatorial types as illustrated in Figure 2.6.
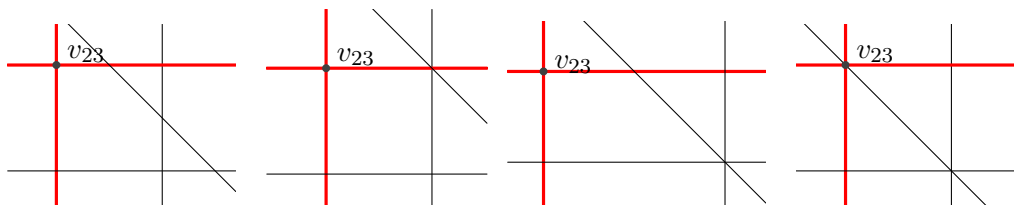

Figure 2.6: Different supporting hyperplane arrangements of halfspaces that fixes hyperplanes H_2 and H_3

For example some of the points from the intersection $C(A, I_{145}) \cap C(A, I_{134})$ give the halfspace arrangement like in Figure 2.7. Intersecting some of the cones of nonempty polytopes reduces the number of different combinatorial types we can obtain.
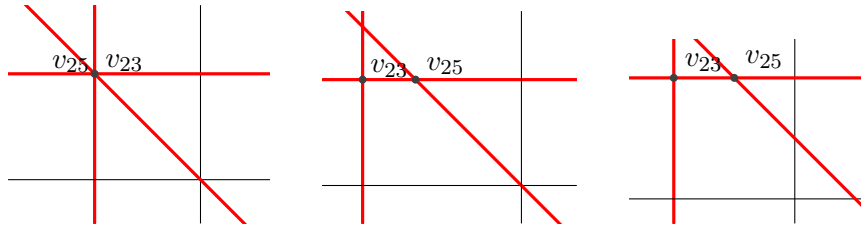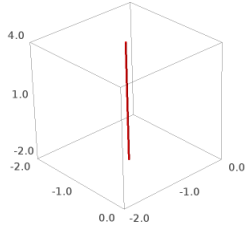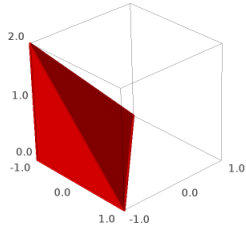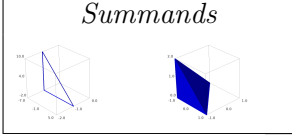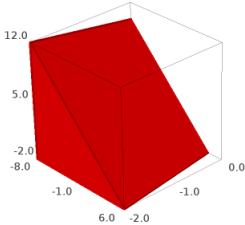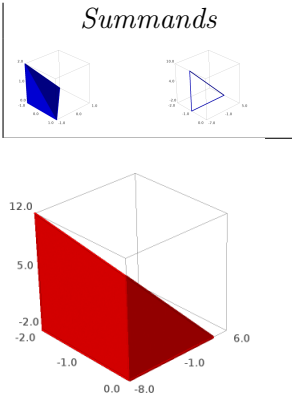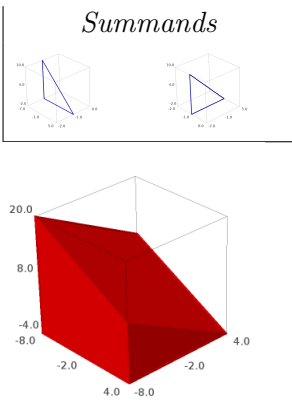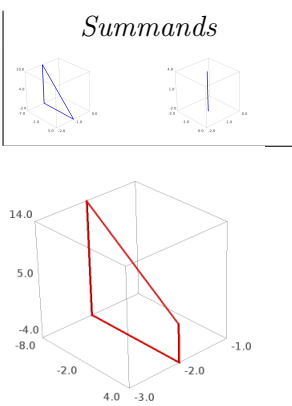
Figure 2.7: Arrangement of supporting hyperplanes of the halfspaces given by some points form $C(A, I_{145}) \cap C(A, I_{134})$
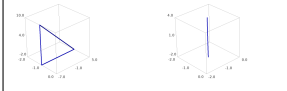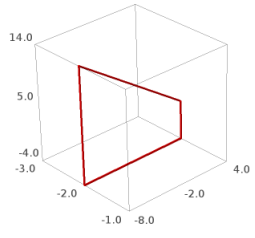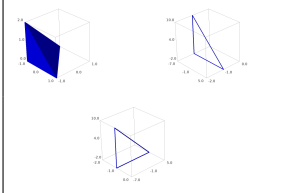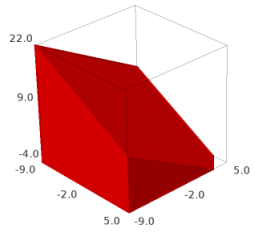
Example 2.2.2.3. *In this example we provide list of chambers of the chamber complex of different matrices computed by the algoritm* 2.1*.*

$$\text{Consider matrix } A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Chambers that the chamber complex of A has	A polytope P_b for b taken from the interior of the chamber	Vertices of P_b
Chamber I: A 1-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 1 ray. **Rays:**$[5, 7, -1, 1, 2, 2]$		Vertices: $[5, -1, -2]$ $[-7, -1, -2]$ $[-7, -1, 10]$
Chamber II: A 1-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 1 ray. **Rays:**$[-1, 1, 5, 7, 2, 2]$		Vertices: $[-1, -7, -2]$ $[-1, -7, 10]$ $[-1, 5, -2]$

Chamber III *A 1-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 1 ray.* **Rays:**$[-1, 1, -1, 1, 2, 2]$		*Vertices:* $[-1, -1, 4]$ $[-1, -1, -2]$
Chamber IV: *A 1-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 1 ray.* **Rays:**$[1, 1, 1, 1, 0, 0]$		*Vertices:* $[-1, 1, 0]$ $[1, -1, 0]$ $[-1, -1, 2]$ $[-1, -1, 0]$
Chamber V: *A 2-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 2 ray.* **Rays:** $[5, 7, -1, 1, 2, 2]$, $[1, 1, 1, 1, 0, 0]$	*Summands*	*Vertices:* $[4, 0, -2],$ $[6, -2, -2],$ $[-8, -2, 12],$ $[-8, -2, -2],$ $[-8, 0, -2]$ $[-8, 0, 10]$

Chamber VI: *A 2-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 2 ray.* **Rays:** $[1,1,1,1,0,0]$, $[-1,1,5,7,2,2]$	*Summands*	*Vertices:* $[-2,6,-2]$, $[0,-8,10]$, $[0,4,-2]$, $[0,-8,-2]$, $[-2,-8,-2]$ $[-2,-8,12]$,
Chamber VII: *A 2-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 2 ray.* **Rays:** $[5,7,-1,1,2,2]$, $[-1,1,5,7,2,2]$	*Summands*	*Vertices:* $[4,-8,-4]$, $[4,-8,8]$, $[4,4,-4]$, $[-8,4,-4]$, $[-8,4,8]$ $[-8,-8,-4]$, $[-8,-8,20]$
Chamber VIII: *A 2-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 2 ray.* **Rays:** $[5,7,-1,1,2,2]$, $[-1,1,-1,1,2,2]$	*Summands*	$[4,-2,2]$, $[4,-2,-4]$, $[-8,-2,-4]$, $[-8,-2,14]$

Chamber IX: *A 2-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 2 ray.* ***Rays:*** $[-1,1,5,7,2,2]$, $[-1,1,-1,1,2,2]$	*Summands*	*Vertices:* $[-2,-8,14]$, $[-2,-8,-4]$, $[-2,4,2]$, $[-2,4,-4]$
Chamber X: *A 3-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 3 ray.* ***Rays:*** $[1,1,1,1,0,0]$, $[5,7,-1,1,2,2]$, $[-1,1,5,7,2,2]$	*Summands*	*Vertices:* $[5,-9,-4]$, $[5,-9,8]$, $[5,3,-4]$, $[3,5,-4]$, $[-9,5,8]$ $[-9,5,-4]$, $[-9,-9,-4]$, $[-9,-9,22]$

	Summands	*Vertices:*
Chamber XI: *A 3-dimensional polyhedron in $\mathbb{Z}^6$ defined as the convex hull of 1 vertex and 3 ray.* ***Rays:*** $[5, 7, -1, 1, 2, 2]$, $[-1, 1, 5, 7, 2, 2]$, $[-1, 1, -1, 1, 2, 2]$		$[3, -9, 12]$, $[3, -9, -6]$, $[3, 3, 0]$, $[3, 3, -6]$, $[-9, 3, -6]$ $[-9, 3, 12]$, $[-9, -9, 24]$, $[-9, -9, -6]$
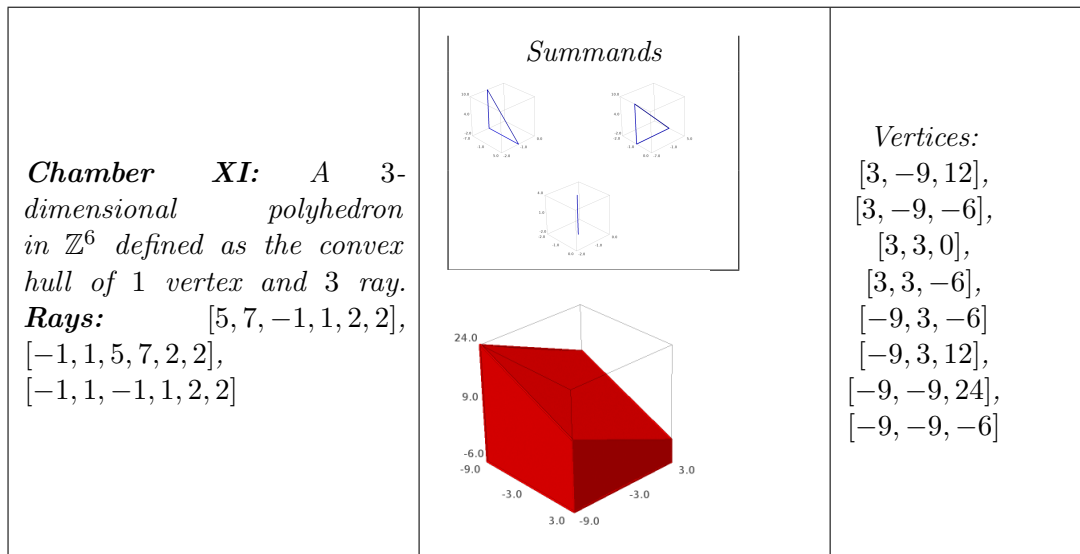
$$\text{For the matrix } A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

the code is too slow to compute the chamber complex.

The algorithm is very slow for matrices in $\mathbb{R}^{m \times 3}$ having more than **7** rows as for the matrix A given at the end of the example 2.2.2.3. As we mentioned before, when there is only one top dimensional chamber in the chamber complex, one can use the existing algorithm from previous works to compute the chamber of a polytope P_b for some b taken form the top dimensional chamber, and take the closure of it. However, when the chamber complex has more than one top dimensional chambers, the algorithm we have can be used. Being able to answer the following question would be helpful to decide which algorithm we use to compute the chamber complex end.

Question 2.2.2.4. *For a given matrix $A \in \mathbb{R}^{m \times d}$, what is the number of top dimensional chambers that the chamber complex of A has?*

In the process of answering this question, and understanding the structure of the chamber complex better, we have the following approach given in the next chapter.

Chapter 3

Toric Varieties and the Chamber Complex

In the previous section, we defined the chamber complex which is the collection of chambers that one can obtain for a given matrix $A \in \mathbb{R}^{m \times d}$. We also know that for each combinatorial type one can obtain by moving the halfspaces given by the matrix A, we have a chamber associated to it such that for each point c, taken from the same chamber, the polytope P_c have the same normal fan. In other words, for each chamber in the chamber complex, there is a normal fan associated to it.

In this section, for each chamber of a given chamber complex, we consider the toric variety of its associated normal fan in order to understand the structure of the chamber complex better.

Recall that the normal fan $N(P)$ of a polytope P is the collection of the normal cones $N(F, P)$ of faces F of the polytope. Moreover, the normal fan $N(P)$ is a polyhedral complex such that the intersection of two cones σ_1, σ_2 is either the empty set or a face of σ_1 and σ_2.

Given a matrix $A \in \mathbb{R}^{m \times d}$, each non-empty polytope we can obtain by inserting a vector $b \in \mathbb{R}^m$ and considering the solution set of $Ax \leq b$ are defined by the half spaces defined by the outer vectors given by the rows of A. The change of the combinatorial type of the polytopes we can obtain occurs when some half spaces get redundant. Having a redundant half space means that its outer vector does not generate a one dimensional cone in the normal fan of the polytope.

Now, let us consider the chamber complex of $A \in \mathbb{R}^{m \times d}$. When two chambers C_1 and C_2 of the chamber complex are neighbors such that they have a common face C_3, it means that polytopes $P_{c_1} = \left\{ x \in \mathbb{R}^d : Ax \leq c_1 \right\}$ for a taken $c_1 \in C_1$, and $P_{c_2} = \left\{ x \in \mathbb{R}^d : Ax \leq c_1 \right\}$ for a taken $c_2 \in C_2$ have a common Minkowski summand that has the same normal fan as the associated normal fan of C_3. In other words, the normal fans $N(P_{c_1})$ and $N(P_{c_2})$ have some common one dimensional cones as, the set of the outer normals of a Minkowski sum are the union of the set of the outer normals of the summands.Please note that they might also have common higher dimensional cones. For the toric variety perspective, it means that the toric varieties $X_{N(P_{c_1})}$ and $X_{N(P_{c_2})}$ have some common open subsets, where $N(P_{c_1})$ and $N(P_{c_2})$ are the associated normal fans of the chambers C_1 and C_2 respectively.

Please note that, when we consider the toric vairety of the normal fan associated to a

chamber C, we consider the fan in the space where it is complete, i.e., the normal fan spans the space.

Let us show the change of the toric varieties of the associated chambers of a chamber complex on the following example.

Example 3.0.0.1. *Consider the chamber complex of the matrix* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$.

The chamber complex of A was presented in Example 2.2.0.3. In this example we will also consider the associated normal fans of the chambers and their toric varieties.

In figure 3.1, you can see the illustration of the chambers in the chamber complex and an example of polytopes one can obtain by taking points form the interiors of the chambers, and the corresponding normal fans.
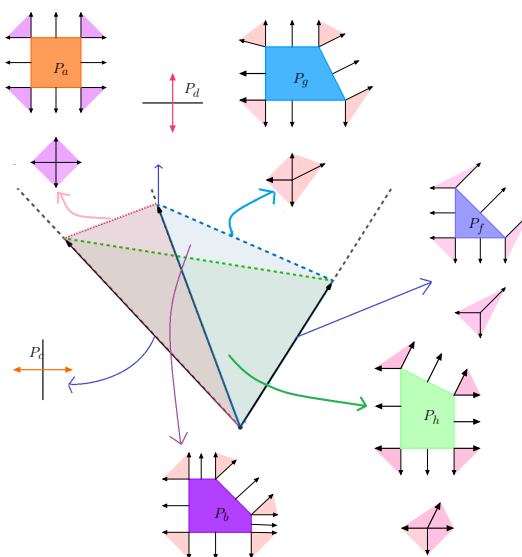

Figure 3.1: Chambers in the chamber complex and the corresponding normal fans

In this example, for each normal fan associated to the chamber complex, we are going to present the toric varieties of the normal fans.

- *Consider the normal fan $N(P_a)$ of the polytope P_a such that this normal fan is associated to the chamber of P_a.*
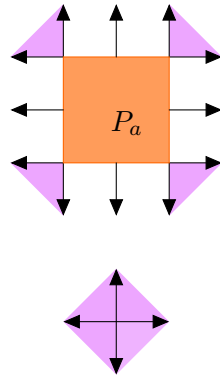
Cones	Generators
σ_1	$e_1 = (1,0)$
σ_2	$e_2 = (0,1)$
σ_3	$-e_1 = (-1,0)$
σ_4	$-e_2 = (0,-1)$
σ_{12}	$e_1 = (1,0), e_2 = (0,1)$
σ_{23}	$e_2 = (0,1), -e_1 = (-1,0)$
σ_{34}	$-e_1 = (-1,0), -e_2 = (0,-1)$
σ_{14}	$e_1 = (1,0), -e_2 = (0,-1)$

From Example 1.2.1.3, we know that the toric variety $X_{N(P_a)} = \mathbb{P}^1 \times \mathbb{P}^1$ which is also the associated variety of the chamber C_{P_a}.

- *Consider the normal fan $N(P_d)$ of the poytope P_d that is the normal fan associated to the chamber C_{P_d}. Please also note that the chamber C_{P_d} is a face of the chamber C_{P_a}.*

Cones	Generators
σ_1	$e_2 = (0,1)$
σ_2	$-e_2 = (0,-1)$
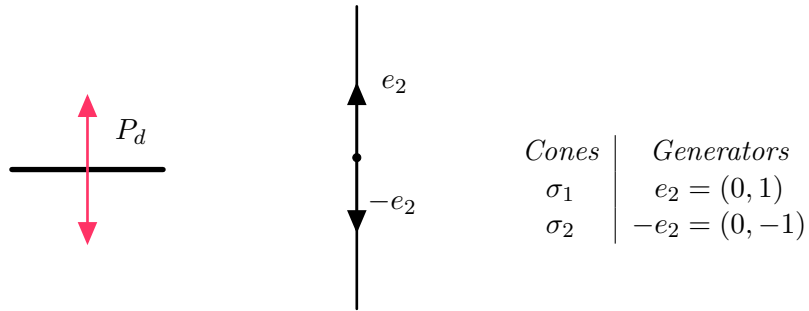
The toric variety of the associated fan and the associated chamber C_{P_d} is

$$X_{N(P_d)} = \mathbb{P}^1.$$

- *Consider the polytope P_c and it is normal fan $N(P_c)$ that are associated to the chamber C_{P_c}.*

Cones	Generators
σ_1	$e_1 = (1,0)$
σ_2	$-e_1 = (-1,0)$
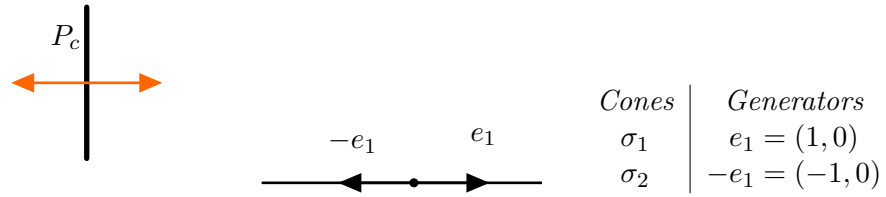
The toric variety of the fan N_{P_c} that is associated toric variety of the chamber C_{P_c} is $X_{N(P_c)} = \mathbb{P}^1$.

- Consider the normal fan $N(P_f)$ of the polytope P_f. $N(P_f)$ is also the normal fan associated to the chamber C_{P_f}.

Cones	Generators
σ_1	$e_1 + e_2 = (1,1)$
σ_2	$-e_1 = (-1,0)$
σ_3	$-e_2 = (0,-1)$
σ_{12}	$e_1 + e_2 = (1,1), -e_1 = (-1,0)$
σ_{13}	$e_1 + e_2 = (1,1), -e_2 = (0,-1)$
σ_{23}	$-e_1 = (-1,0), -e_2 = (0,-1)$
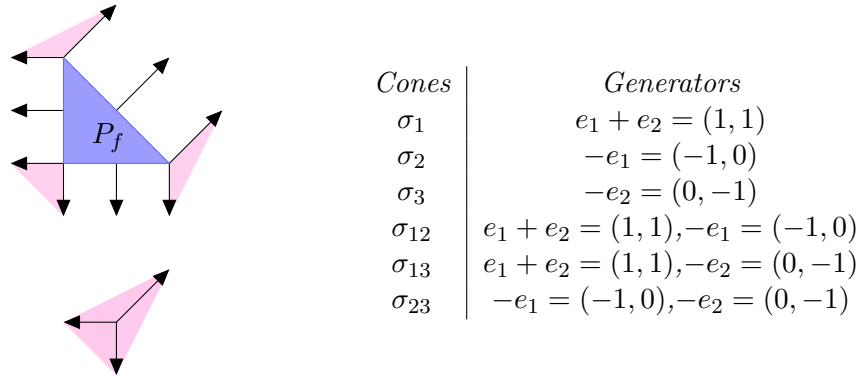
Since the normal fan $N(P_f)$ is the same as the normal fan $N(\Delta_2)$ where Δ_2 is the 2 dimensional simplex, by the remark 1.2.1.7, the toric variety of $N(P_f)$ is $X_{N(P_f)} = \mathbb{P}^2$.

- Consider the polytope P_g and its normal fan N_{P_g} that is associated to the chamber C_{P_g}.
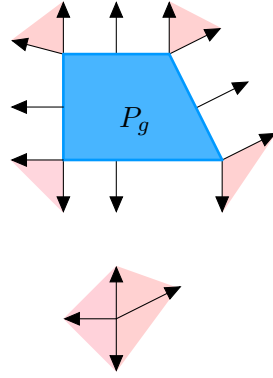
Cones	Generators
σ_1	$e_1 + e_2 = (1,1)$
σ_2	$e_2 = (0,1)$
σ_3	$-e_1 = (-1,0)$
σ_4	$-e_2 = (0,-1)$
σ_{12}	$e_1 + e_2 = (1,1), e_2 = (0,1)$
σ_{14}	$e_1 + e_2 = (1,1), -e_2 = (0,-1)$
σ_{23}	$e_2 = (0,1),\ -e_1 = (-1,0)$
σ_{34}	$-e_1 = (-1,0), -e_2 = (0,-1)$

The normal fan $N(P_g)$ is the star subdivision of the fan $N(P_f)$ along $\sigma_{12} \in N(P_f)$ by Definition *1.2.1.8* as the generator of $\sigma_2 \in N(P_g)$ is the sum of the generators of $\sigma_{12} \in N(P_f)$, and the cones $\sigma_{12}, \sigma_{23} \in N(P_g)$ gives a subdivision of $\sigma_{12} \in N(P_f)$.

Thus the toric variety $X_{N(P_g)}$ is the blow up of $\mathbb{P}^2$ at one point by Proposition *1.2.1.11*. One can also observe that the fan $N(P_g)$ is a rotated version of the normal fan of the Hirzebruch surface H_1.
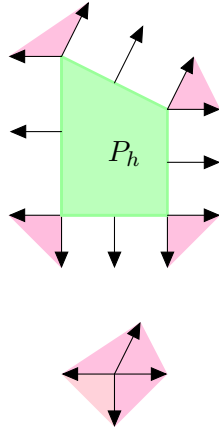
- Consider the normal fan N_{P_h} of the polytope P_h that is the associated normal fan of the chamber C_{P_h}.

Cones	Generators
σ_1	$e_1 + e_2 = (1,1)$
σ_2	$e_1 = (1,0)$
σ_3	$-e_1 = (-1,0)$
σ_4	$-e_2 = (0,-1)$
σ_{12}	$e_1 + e_2 = (1,1), e_1 = (1,0)$
σ_{14}	$e_1 + e_2 = (1,1), -e_2 = (0,-1)$
σ_{24}	$e_1 = (1,0),\ -e_2 = (0,-1)$
σ_{34}	$-e_1 = (-1,0), -e_2 = (0,-1)$

The normal fan $N(P_h)$ is the star subdivision of the fan $N(P_f)$ along $\sigma_{13} \in N(P_f)$ by Definition *1.2.1.8*. Thus, the toric variety $X_{N(P_h)}$ that is associated to the chamber C_{P_h} is the blow up of $\mathbb{P}^2$ at one point by Proposition *1.2.1.11*. When we consider

$N(P_h)$, we also see that it is a rotated version of the normal fan of the Hirzebruch surface H_1.

- Consider the polytope P_b and its normal fan $N(P_b)$ that is associated to the chamber C_{P_b}.

Cones	Generators
σ_1	$e_1 + e_2 = (1,1)$
σ_2	$e_1 = (1,0)$
σ_3	$-e_1 = (-1,0)$
σ_4	$-e_2 = (0,-1)$
σ_5	$e_2 = (0,1)$
σ_{15}	$e_1 + e_2 = (1,1), e_2 = (0,1)$
σ_{12}	$e_1 + e_2 = (1,1), e_1 = (1,0)$
σ_{24}	$e_1 = (1,0),\ -e_2 = (0,-1)$
σ_{34}	$-e_1 = (-1,0),\ -e_2 = (0,-1)$
σ_{35}	$-e_1 = (-1,0), e_2 = (0,1)$
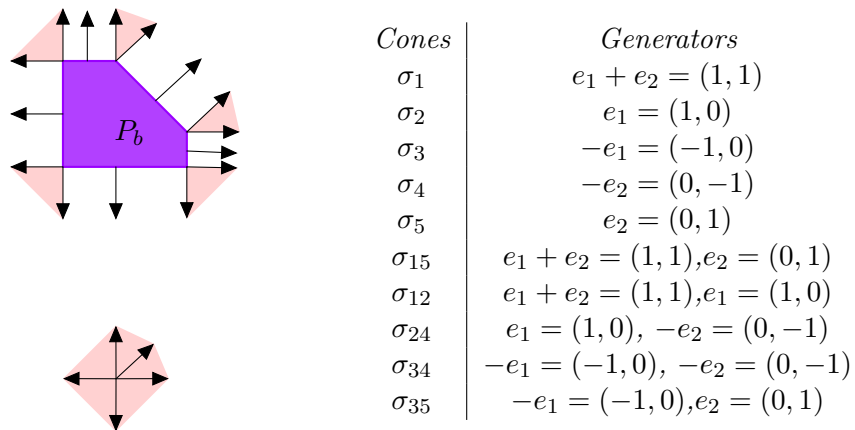
The fan $N(P_b)$ is the star subdivision of the fan $N(P_a)$ along the cone $\sigma_{12} \in N(P_a)$ by Definition 1.2.1.8. Thus the toric variety $X_{N(P_b)}$ associated to the chamber C_{P_b} is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point.
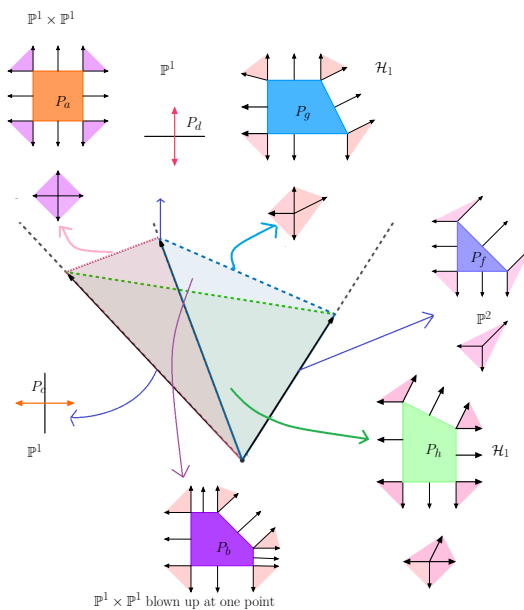

Figure 3.2: The chamber complex with the normal fans and toric varieties associated to the chambers

When we consider the chamber C_{P_a} in Figure 3.2, we see that its closure has faces C_{P_d} and C_{P_c}. While the toric varieties $X_{N(P_c)}$ and $X_{(N(P_d))}$ are $\mathbb{P}^1$, the toric variety $X_{N(P_a)}$ is $\mathbb{P}^1 \times \mathbb{P}^1$, which also corresponds to the normal fan of the Minkowski sum of the polytopes having normal fans $N(P_c)$ and $N(P_d)$. The reason is that the polytopes P_d and P_c are orthogonal so that their normal fans and we have $N(P_a) = N(P_c) \times N(P_d)$ as the normal fan of the sum. By Proposition 1.2.1.4, the toric variety of $N(P_c) \times N(P_d)$ is $X_{N(P_c)} \times X_{N(P_d)}$.

When we consider the chamber C_{P_g} in Figure 3.2, we see that its closure has faces C_{P_d} and C_{P_f} and their associated toric varieties are $X_{N(P_d)} = \mathbb{P}^1$ and $X_{N(P_f)} = \mathbb{P}^2$. Moreover, the toric variety $X_{N(P_g)}$, that is the toric variety of the normal fan of the Minkowski sum of P_d and P_f, is the blow up of $\mathbb{P}^2$ at one point since $N(P_g)$ is a star subdivision of $N(P_f)$.

Now, consider the top dimensional chamber C_{P_b} of the chamber complex. The corresponding toric variety is the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point which can also be expressed as blow up of H_1 at one point as $N(P_b)$ is a star subdivision of $N(P_g)$ along the cone $\sigma_{14} \in N(P_g)$.

Theorem 3.0.0.2. *Let $A \in \mathbb{R}^{m \times d}$ be a given matrix of full rank with $m > d$. If there exists $a, b, c \in \mathbb{R}^m$ with nonempty $P_a \subset \mathbb{R}^k \times \{0\} \subset \mathbb{R}^d, P_b \subset \{0\} \times \mathbb{R}^{d-k} \subset \mathbb{R}^d$, and $P_c = P_a + P_b$, then the toric variety associated to the chamber $C_{(P_c)}$ of the chamber complex of A is*

$$X_{N(P_c)} = X_{N(P_a)} \times X_{N(P_b)}$$

where $X_{N(P_a)}$ and $X_{N(P_b)}$ are toric varieties associated to the chambers C_{P_a}, C_{P_b} respectively.

Proof. Since there exist points $a, b \in \mathbb{R}^m$ such that the polytopes P_a and P_b live in orthogonal subspaces of $\mathbb{R}^d$, the matrix A is of the form

$$A = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \\ C & D \end{bmatrix}$$

for some matrices B_1, B_2, C, D such that, some polytopes we can obtain by moving the half spaces that A defines will have orthogonal one dimensional cones in their normal fans.

Moreover, the fact that there exist points $a, b \in \mathbb{R}^m$ which give such non-empty polytopes P_a, P_b means that the chamber complex C_A of A has the chamber C_{P_a} that is the set of points $v \in \mathbb{R}^m$ for which polytopes P_v have the same normal fan as P_a, and the chamber C_{P_b} that is the set of points $w \in \mathbb{R}^m$ for which polytopes P_w have the same normal fan as P_b.

Moreover, the assumption that there is a vector $c \in \mathbb{R}^m$ such that $P_c = P_a + P_b$ says that the chamber complex C_A has the chamber C_{P_c} such that the closure of C_{P_c} contains the chambers C_{P_a} and C_{P_b} as P_a and P_b are the Minkowski summands of P_c.

By the assumption, there are polytopes P_a and P_b we can obtain that are living in the orthogonal spaces so that, the outer normal vectors of their defining half spaces live in the

orthogonal spaces. Thus, for such P_a and P_b, $N(P_a)$ and $N(P_b)$ also live in the orthogonal spaces. Moreover, as the points $x \in P_a$ are of the form $x = (x_1, \ldots, x_k, 0, \ldots, 0)$ and points $y \in P_b$ are of the form $(0, \ldots, 0, y_1, \ldots, y_l)$, their Minkowski sum is

$$P_c = P_a + P_b = \{x + y : x \in P_a, y \in P_b\} = P_a \times P_b.$$

Vertices of P_c are pairs (v_i, v_j) where v_i is a vertex of P_a and w_j is a vertex of P_b. We also know that the set of one dimensional cones of $N(P_c)$ is the union of the set of one dimensional cones of $N(P_a)$ and $N(P_b)$, and for each vertex (v_i, w_j), $N(P_c)$ will contain the face cone generated by $gens(N(v_i, P_a)), gens(N(w_j, P_b))$ such that

$$N(P_a + P_b) = N(P_c) = N(P_a) \times N(P_b) \subset \mathbb{R}^k \times \mathbb{R}^l.$$

Recall that $N(P_a)$ is the normal fan associated to C_{P_a}, $N(P_b)$ is the normal fan associated to C_{P_b} and $N(P_c = P_a + P_b)$ is the associated normal fan of C_{P_c}. Thus by Proposition 1.2.1.4,

$$X_{N(P_c)} = X_{N(P_a)} \times X_{N(P_b)}.$$

$\square$

Theorem 3.0.0.3. *Let $A \in \mathbb{R}^{m \times d}$ be a given matrix. If there exist points $a, b, c \in \mathbb{R}^m$ such that $P_a = \Delta_k, P_b = \Delta_l \subset \mathbb{R}^d$, $k \geq 8, l = \frac{k}{2} - 3 \in \mathbb{N}_{>0}$, and $P_c = P_a + P_c \neq P_a \times P_b$, then the associated toric variety $X_{N(P_c)}$ of the chamber C_{P_c} is the blow up of $\mathbb{P}^k$ at one point.*

Proof. Since there exists $a, b, c \in \mathbb{R}^m$ such that $P_a = \Delta_k$ and $P_b = \Delta_l$ are nonempty and $P_c = P_a + P_b$, the chamber complex C_A of A has the chambers $C_{P_a}, C_{P_b}, C_{P_c}$. Furthermore, since $P_c = P_a + P_b \neq P_a \times P_b$, the polytopes $P_a = \Delta_k$ and $P_b = \Delta_l$ are not orthogonal. Recall that a standard n-simplex Δ_n has the outer normal vectors that are the rows of the following matrix:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ -1 & -1 & \cdots & -1 \end{bmatrix}_{(n+1) \times n}.$$

Thus, the matrix A has the following sub-matrices:

$$B_1 = \begin{bmatrix} I(k) & 0 \\ -1 & 0 \end{bmatrix}_{(k+1) \times d} \quad \text{and } B_2 = \begin{bmatrix} I(l) & 0 \\ -1 & 0 \end{bmatrix}_{(l+1) \times d}$$

where $I(i) \in \mathbb{R}^{i \times i}$ where $i = k, l$ is the identity matrix and as $k > l$, $l = d - 2$, B_2 is a sub-matrix of B_1 . Each row of the matrix B_1 is the generator of a one dimensional cone of the normal fan $N(P_a)$ and each row of the matrix B_2 is the generator of a one dimensional cone of $N(P_b)$.

Since the normal fan the Minkowski sum of two polytopes is a common refinement of normal fan of summands, each row of the matrix B_2 and the last row of B_1 is a generator of a one dimensional cone of $N(P_c)$.

Recall that for an n-simplex Δ_n where $n > 2$, the intersection of the half space H, whose outer normal vector is $(-1, \ldots, -1) \in \mathbb{R}^n$, with any $n-2$ hyperplanes that have rows of $I(n) \in \mathbb{R}^{n \times n}$ as outer normal vectors give a vertex of Δ_n (not all vertices). This means that for each $n-2$ subset of $I(n) \in \mathbb{R}^{n \times n}$, the normal fan of Δ_n contains cones generated by $\{k, (-1, \ldots, -1) : k \in K, (-1, \ldots, -1) \in \mathbb{R}^n\}$.

Thus, for each $k-2$ subset K of $I(k)$, the normal fan $N(P_c = \Delta_k + \Delta_l)$ contains cones generated by

$$\{(k,0), \beta_1 : k \in K, (k,0) \in B_1 \, \beta_1 = (-1, \ldots, -1, 0, \ldots, 0) \in B_1\}.$$

Moreover the one dimensional cone generated by $\beta_2 = (-1, \ldots, -1, 0, \ldots, 0) \in B_2$ is also contained in $N(P_c)$, and $\beta_2 = \beta_1 + \sum_{i=l+1}^{k} B_1(i)$ where $B_1(i)$ is the ith row of B_1, it gives a star sub-division of $N(P_a)$ along the cone

$$\sigma = Pos(\beta_1, B_1(i) : l = \frac{k}{2} - 3 < i \leq k) \in N(P_a)$$

by Definition 1.2.1.8.

Since $N(P_c) = \Delta_k + \Delta_l$ is a star subdivision of $N(P_a)$ along

$$\sigma = Pos(\beta_1, B_1(i) : l = \frac{k}{2} - 3 < i \leq k) \in N(P_a),$$

and $X_{N(P_a)} = \mathbb{P}^k$, $X_{N(P_c)}$ is a blow up of $\mathbb{P}^k$ at one point by Proposition 1.2.1.11. $\qquad \square$

3.0.1 Toric Variety of the Chamber Complex

In this section, we consider the toric variety of a given chamber complex instead of the toric variety of the normal fans associated to each chamber.

By definition, the chambers of a given chamber complex have no lines going through them such that they are strictly convex polyhedral cones. Moreover, any chamber complex is a polyhedral complex, i.e., the satisfy the conditions of the definition of a fan. Thus we can mention the toric variety of the chamber complex which is a fan such that the cones of the fan are chambers.

Remark 3.0.1.1 (Cox, Little, Schenk [4])**.** *Given a toric variety X_Σ of a fan Σ, the number of fixed points of the action $\gamma : T \times X_\Sigma \to X_\Sigma$ is the number of top dimensional cones of the fan Σ.*

Theorem 3.0.1.2. *Let C_A be the chamber complex of a given matrix $A \in \mathbb{R}^{m \times d}$. The number of top dimensional chambers of C_A is the number of fixed points of the natural action $\gamma : T \times X_{C_A} \to X_{C_A}$.*

Proof. As we mentioned before, by the definition, a chamber complex is a polyhedral complex and each chamber it contains is a strictly convex rational cone so that the chamber complex is a fan. By Remark 3.0.1.1, The number of top dimensional chambers that C_A has is the number of fixed points of the torus action on the variety X_{C_A}. $\square$

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